Signal Jamming in a Sequential Auction☆

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Abstract

In a sequence of first-price auctions with stable private values bidders strategically conceal their private information until the last auction. We characterize equilibrium bidding and explore how such signal jamming affects the dynamics of equilibrium prices.

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1. Introduction

Many market transactions have an auction structure, and many such auctions are recurring events. For example, price competition between retailers is essentially a (reverse) auction. And this auction is typically a recurring event in which the relevant valuations (unit costs) are stable, at least for some time. Evidently, in this case bidders must pay attention to the information they reveal about their valuations through their bids. This gives rise to a problem of strategic information transmission.

The present paper analyzes equilibrium bidding in a sequence of first-price auctions, when bidders have stable private values. Bidders want to win each auction, but they are also concerned with concealing their valuation in order to reduce the intensity of price competition in later auctions.

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If bidders play strictly monotone strategies in the first auction, they reveal their private information, and the second auction is one under complete information, resulting in fierce competition that wipes out profits. Bidders thus attempt to keep their rival unsure about their valuation by playing non-monotone (partial-pooling) strategies, mimicking the bidder with a low valuation with some probability.

Partial-pooling results have been observed in other branches of the auctions literature. For example, Haile (2000) finds that the possibility of resale, triggered by more precise information acquired after the auction, induces pooling for a range of low signals at a bid equal to the reserve price.

More in line with our focus on strategic information transmission is the analysis of sequential procurement by Waehrer (1999) and Kannan (2010), and the analysis of repeated contests by Münster (2009). There, pooling also serves the purpose to blur the information revealed through bids to prevent that the procurer resp. other contestants take(s) advantage of it.

2. The model

Consider a sequence of two first-price auctions for two identical objects, and two ex ante symmetric bidders, named 1 and 2. Bidders draw their private valuation before the first auction and keep that valuation in the second auction. Before the second object is auctioned, bidders observe both bids of the first auction and use this information to update their beliefs concerning their rival's valuation. Valuations V are *iid* random variables which assume either a low value 0 (normalized) or a high value $v > 0$ with the prior probability $\rho := \Pr\{V = v\} \in (0, 1)$.

If bidders tie, the winner is selected by flipping a fair coin, with one exception: If bidders tie in the second auction and exactly one bid was positive in the first auction, the one who made a positive bid in the first auction is selected as the winner.[1]

We denote bidder i's bid in the jth auction by b_i^j, and continuation payoffs by $\pi(h)$, where h denotes the history of the game prior to the second auction.

The possible histories of the game, h, are described by past bids observed by both players. The following histories must be distinguished; there, only the sign of observed bids matters: 1) The history at the beginning of the game, $h_\emptyset$; 2) the histories with both bids either zero or positive: $h_{00} := \{b_1^1 = 0, b_2^1 = 0\}$ and

[1]Without this assumption the existence of equilibrium fails. However, this problem is artificial since existence can be restored by allowing positive but infinitesimally small bids. Therefore, it is appropriate to bypass this problem by a convenient tie rule, as proposed here.

$h_{11} := \{b_1^1 > 0, b_2^1 > 0\}$; 3) the histories with one positive bid and one bid equal to zero: $h_{10} := \{b_1^1 > 0, b_2^1 = 0\}$ and $h_{01} := \{b_1^1 = 0, b_2^1 > 0\}$.

3. Equilibrium

A bidder with valuation $V = 0$ obviously bids zero with certainty in both auctions (which will not be repeated from here onwards). This does not, however, imply that a zero bid can only come from a bidder with valuation $V = 0$. Indeed, in a signal-jamming equilibrium a bidder with $V = v$ may also bid zero in the first auction in order to keep his rival in doubt about his valuation.

We now solve the equilibrium strategies of a bidder with valuation $V = v$ in both auctions, for all possible histories of the game, employing the equilibrium concept of a sequential equilibrium with observable moves. Thereby, mixed strategies are stated as cumulative distribution functions.

As a working hypothesis, suppose $F : [0, \bar{b}] \to [0, 1]$ is the symmetric mixed-strategy equilibrium of a bidder with $V = v$ in the first auction (history $h_\emptyset$), where $F(0)$ may be positive. This allows us to characterize the equilibrium play in the second auction. We then use this to compute the symmetric equilibrium of the first auction. Altogether, this procedure confirms that the game has a unique symmetric equilibrium and solves it explicitly.

To avoid unnecessary duplication we state only the equilibrium strategies and beliefs of one player, named player 1.

Equilibrium in the second auction. After the first auction, bidders observe bids, process this information to update their beliefs about the rival's valuation, and then play the second auction. Updated beliefs must be consistent with the equilibrium strategy of the first auction and the observed bids. Hence, using Bayes's rule whenever applicable, posterior beliefs are:[2]

$$\Pr\{V_2 = v \mid b_2^1\} = \begin{cases} 1 & \text{if} \quad b_2^1 > 0, \\ \frac{F(0)\rho}{F(0)\rho + (1-\rho)} =: q & \text{if} \quad b_2^1 = 0. \end{cases} \tag{1}$$

Proposition 1 (Second auction)**.** *Consider the second auction. The equilibrium strategy of player 1 (with $V_1 = v$) depends on the history as follows: $h_{11} \Rightarrow b_1^2 = v$,*

[2]Note, this belief system involves only a fairly innocent prescription of "off-equilibrium path" beliefs by stipulating that $\Pr\{V_2 = v \mid b_2^1 > 0\} = 1$ also for bids that are higher than "predicted", i.e., for $b_2^1 > \bar{b}$.

$h \in \{h_{00}, h_{01}\} \Rightarrow G(b_1^2)$, $h = h_{10} \Rightarrow H(b_1^2)$:

$$G : [0, qv] \to [0, 1], \quad G(b) = \frac{b(1-q)}{(v-b)q} = \frac{b(1-\rho)}{(v-b)\rho F(0)}, \tag{2}$$

$$H : [0, qv] \to [0, 1], \quad H(b) = \frac{v(1-q)}{v-b} = \frac{v(1-\rho)}{(v-b)\,(1-\rho+\rho F(0))}, \tag{3}$$

where H has one mass point at $b = 0$. The associated equilibrium continuation payoffs are $\pi(h_{11}) = 0, \pi(h_{00}) = \pi(h_{10}) = \pi(h_{01}) = v(1-q)$.

Proof. Equilibrium strategies and payoffs are self-evident for h_{11}. Histories h_{01} and h_{10} result in an asymmetric, and history h_{00} in a symmetric one-shot auction, solved in Jeitschko and Wolfstetter (2002, Prop. 3). □

Equilibrium in the first auction. Signal jamming pays for bidder 1 only if bidder 2 also has a high valuation. However, signal jamming is costly, and its benefit outweighs the cost only if it is sufficiently likely that bidder 1 meets a high-value bidder 2.

Proposition 2. *Suppose $\rho > 1/3$. The equilibrium strategy in the first auction (conditional on $V = v$) is $F : [0, \bar{b}] \to [0, 1]$:*

$$F(b) := \frac{1-\rho}{\rho}\frac{b}{v-b} + \frac{v}{v-b}\frac{\sqrt{3-4\rho+\rho^2} - 2(1-\rho)}{\rho}, \tag{4}$$

$$\bar{b} := v\Big(2 - \rho - \sqrt{3-4\rho+\rho^2}\Big). \tag{5}$$

F has a mass point at zero: $F(0) = \sqrt{(1/\rho - 1)(3/\rho - 1)} - 2(1/\rho - 1) > 0$ that has a maximum at $\rho = 3/4$ and approaches zero as $\rho \to 1$ and as $\rho \to 1/3$.

Proof. Consider one bidder, say bidder 1 with $V = v$, and history $h_\emptyset$ (first auction). To confirm the asserted equilibrium mixed-strategy F, stated in (4), we must show that this bidder is indifferent between all bids from the support of F, which is $[0, \bar{b}]$ where $\bar{b}$ is stated in (5).

If bidder 1 with $V = v$ makes a bid $b \in (0, \bar{b}]$, his payoff is equal to

$$(v - b + \pi(h_{11}))\,\rho\,(F(b) - F(0)) + (v - b + \pi(h_{10}))\,((1-\rho) + \rho F(0)). \tag{6}$$

Whereas if he bids zero, his payoff is

$$\left(\frac{v}{2} + \pi(h_{00})\right)(\rho F(0) + (1-\rho)) + \pi(h_{01})\rho(1 - F(0)). \tag{7}$$

And the assertion follows immediately. □

We mention that no signal jamming occurs if $\rho \leq {}^1\!/_3$; in that case, a high-value bidder plays the myopic strategy $K : [0, \rho v] \to [0, 1]$, $K(b) := ({}^{1-\rho}\!/_{\rho})({}^{b}\!/_{v-b})$ in the first auction, and in the second auction bids v if he has observed a positive first-auction bid from his rival, and otherwise bids zero. While this is the unique symmetric Bayesian Nash equilibrium, that equilibrium is not a sequential equilibrium.[3]

4. Signal jamming

Signal jamming occurs if a bidder with a high valuation bids zero in the first auction with positive probability, $F(0) > 0$, and thus sometimes mimics a bidder with a low valuation in order to keep the rival uninformed.

Altogether, bidder 1 benefits from signal jamming if and only if bidder 2 also has a high valuation. Signal jamming leads bidder 2 to update his belief from ρ to $\Pr\{V_1 = v \mid b_1^1 = 0\} = q < \rho$ instead of revealing bidder 1's type, which in turn induces him to bid stochastically lower, no matter how he bid in the first auction. If bidder 2 made a positive bid in the first auction, he plays the stochastically lower mixed-strategy $H(b) > G(b), \forall b \in [0, qv]$, and if he also engaged in signal jamming, both bidders play the mixed-strategy G, which preserves a positive expected profit in the second auction.

However, signal jamming is also costly since it entails the risk of losing the first auction. It follows that it pays to "invest" in signal jamming only if it is sufficiently likely that the rival has a high valuation. Interestingly, this relationship is not monotone, and $F(0)$ has a global maximum at $\rho = {}^3\!/_4$.

We mention that signal jamming induces pointwise less aggressive bidding in the first auction, in the sense that the myopic strategy K first-order stochastically dominates the (continuously extended) strategy $\bar{F}(b) := \min\{F(b), 1\}$, i.e., $\bar{F}(b) \geq K(b), \forall\, b \in [0, \rho v]$.

5. Dynamics of equilibrium prices

The study of price sequences in sequential auctions has received much attention in the literature (see, for example, McAfee and Vincent, 1993; Gale and Hausch, 1994; Jeitschko, 1999). In the present context, one might expect prices to

[3] For if we suppose $V_1 = v$ and consider the history h_{00}, the consistency of beliefs requires that each bidder believes that his rival has a zero valuation; but then bidding zero is not the best response of bidder 1 (in fact, no best response exists in that case).

be stochastically increasing since signal jamming involves bidding low in the first auction.

In the following assume $\rho > 1/3$ (because otherwise no signal jamming occurs and equilibrium prices are stationary), and denote the continuously extended strategies for the enlarged domain $[0, v]$ by $\bar{G}(b) := \min\{G(b), 1\}$, $\bar{H}(b) := \min\{H(b), 1\}$.

Lemma 1. *The probability distribution of the equilibrium price in the first auction, $F_{P^1} : [0, v] \to [0, 1]$, is*

$$\begin{aligned} F_{P^1}(p) :&= \Pr\{P^1 \leq p\} \\ &= \Pr\{\tilde{b}_1^1 \leq p \quad and \quad \tilde{b}_2^1 \leq p\} \\ &= \begin{cases} (1-\rho+\rho F(p))^2 & if \quad p \leq \bar{b}, \\ 1 & if \quad p \geq \bar{b}. \end{cases} \end{aligned} \tag{8}$$

F_{P^1} has exactly one mass point, $F_{P^1}(0) = \left((1-\rho) - \sqrt{3-4\rho+\rho^2}\right)^2 > 0$, that is strictly decreasing in ρ with $\lim_{\rho\to 1} F_{P^1}(0) = 0$.

Lemma 2. *The probability distribution of the equilibrium price in the second auction, $F_{P^2} : [0, v] \to [0, 1]$, is*

$$\begin{aligned} F_{P^2}(p) &= \Pr\{P^2 \leq p\} \\ &= \rho^2 \left((1-F(0))^2 \cdot 0 + F(0)^2\bar{G}(p)^2 + 2F(0)(1-F(0))\bar{G}(p)\bar{H}(p)\right) \\ &+ 2\rho(1-\rho)\left(F(0)\bar{G}(p) + (1-F(0))\bar{H}(p)\right) + (1-\rho)^2. \end{aligned} \tag{9}$$

F_{P^2} has mass points at $p = 0$ and $p = v$:

$$F_{P^2}(0) = (1-\rho)\sqrt{3-4\rho+\rho^2} > 0, \tag{10}$$

$$\Pr\{P^2 = v\} = \left(2-\rho-\sqrt{3-4\rho+\rho^2}\right)^2 > 0. \tag{11}$$

$\Pr\{P^2 = v\}$ *is strictly increasing in ρ, and $F_{P^2}(0)$ is decreasing with* $\lim_{\rho\to 1} F_{P^2}(0) = 0$.

Proposition 3. *Equilibrium prices are increasing in the sense of first-order stochastic dominance, $F_{P^2}(p) \leq F_{P^1}(p)$ (with strict inequality except for $p = v$), if and only if ρ is sufficiently large, i.e., $\rho > \rho^* := 2 - 3/5\sqrt{5}$.*

This strong stochastic order does not apply to $\rho \in (1/3, \rho^*)$. However, in either case $E[P^2] > E[P^1]$.

Proof. 1) Suppose $\rho > \rho^*$. Let $p \in [0, v)$ and define (where the functions G and H are applied to the enlarged domain $[0, v)$):

$$\tilde{F}_{P^2}(p) := \rho^2 \left((1 - F(0))^2 \cdot 0 + F(0)^2 G(p)^2 + 2F(0)(1 - F(0))G(p)H(p) \right) + 2\rho(1 - \rho)\left(F(0)G(p) + (1 - F(0))H(p)\right) + (1 - \rho)^2.$$

By definition, $G(p) \geq \bar{G}(p), H(p) \geq \bar{H}(p)$, with equality for all $p \in [0, qv]$ and strict inequality for all $p \in (qv, v)$. Therefore, it follows immediately that $\tilde{F}_{P^2}(p)$ is a pointwise upper bound of $F_{P^2}(p)$, i.e., $\tilde{F}_{P^2}(p) \geq F_{P^2}(p)$, for all $p \in [0, v)$, and hence, in particular, for all $p \in [0, \bar{b}]$.

As one can easily confirm, $F_{P^1} > \tilde{F}_{P^2}(p), \forall p \in [0, \bar{b}] \iff \rho > \rho^*$. Since $F_{P^1}(p) = 1, \forall b \geq \bar{b}$ and $F_{P^2} < 1, \forall b < v$ (since it has a mass point at $p = v$), we conclude that $\rho > \rho^* \Rightarrow F_{P^2}(p) \leq F_{P^1}(p)$, with strict inequality everywhere except at $p = v$, as illustrated in the right-hand side of Figure 1. Of course, the established first-order stochastic-dominance relationship implies $E[P^2] > E[P^1]$.

2) Suppose $\rho \in (1/3, \rho^*)$. Then, as one can easily confirm, $F_{P^2}(0) > F_{P^1}(0)$. Moreover, $F_{P^1}(p) = 1 > F_{P^2}(p), \forall p \in [\bar{b}, v)$ (since $F_{P^1}(\bar{b}) = 1$ and F_{P^2} has a mass point at $p = v$). Therefore, $F_{P^2}(p)$ and $F_{P^1}(p)$ must intersect at least once; hence no first-order stochastic-dominance relationship applies to P^2 and P^1, as illustrated in the left-hand side of Figure 1.

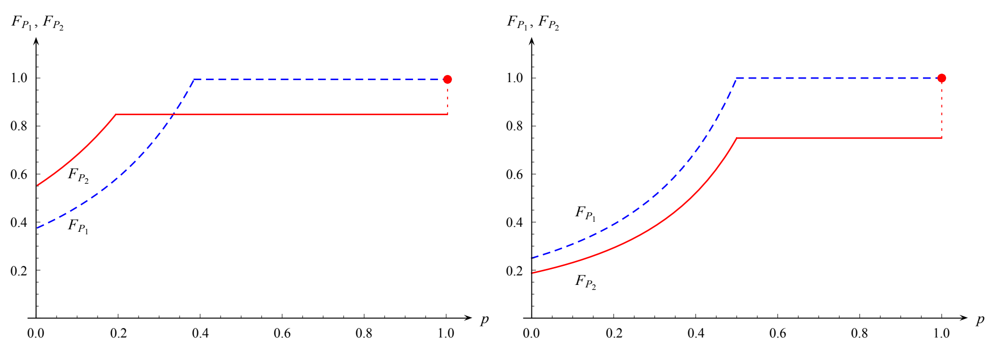

Figure 1: Comparison between F_{P^1} (dashed) and F_{P^2} (solid) for $v = 1$ and $\rho = 1/2 < \rho^*$ (left) resp. $\rho = 3/4 > \rho^*$ (right)

3) Computing expected equilibrium prices, one finds, using (8) and then (4) and (5),

$$E[P^1] = \int_0^v p dF_{P^1}(p) = \int_0^{\bar{b}} 2p\left(1 - \rho + \rho F(p)\right)\rho dF(p) = v\left(7 - 2\rho(4 - \rho) - 2(2 - \rho)\sqrt{3 - 4\rho + \rho^2}\right).$$

Similarly, the expected price in the second auction is, using equations (9) and (11),

$$E[P^2] = \int_0^v p dF_{P^2}(p) = \int_0^{qv} p dF_{P^2}(p) + v \Pr\{P^2 = v\}$$
$$= \frac{v\left(14 - 5\rho^3 + 25\rho^2 - 34\rho - (8 + 5\rho^2 - 14\rho)\sqrt{3 - 4\rho + \rho^2}\right)}{\sqrt{3 - 4\rho + \rho^2} - (1 - \rho)}$$

Therefore, after some rearrangements, for all $\rho > 1/3$:

$$E[P^1] - E[P^2] = v\sqrt{3 - 4\rho + \rho^2}\left(5\rho - 7 + 4\sqrt{3 - 4\rho + \rho^2}\right) < 0,$$

as asserted. □

6. Discussion

In the present paper we assumed that bidders have stable valuations. An alternative framework would be to assume that valuations are subject to stochastic scale effects, as in Jeitschko and Wolfstetter (2002).

We also assumed that bidders observe all first auction bids before they bid in the second auction. If instead bidders could only learn whether they either won or lost the first auction, in some subgames bidders would know the rank order of valuations, as in Landsberger et al. (2001) and Février (2003).

Moreover, we assumed a passive auctioneer. Therefore, signal jamming served exclusively the purpose of misleading the rival bidder. The scope of signal jamming is further increased if the auctioneer is able to adjust reserve prices to take advantage of information acquired during the first auction, as in Caillaud and Mezzetti (2004).

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