Auctions with imperfect commitment when the reserve may signal the cost to re-auction ☆

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ABSTRACT

If bidders are uncertain whether the auctioneer sticks to the announced reserve, they respond with strategic non-participation, speculating that the auctioneer may revoke the reserve. However, the reserve inadvertently signals the auctioneer's type, which drives multiplicity of equilibria. Eliminating belief systems that violate the "intuitive criterion" yields a unique equilibrium reserve price equal to the seller's consumption value. Paradoxically, even if bidders initially believe that the auctioneer is bound by his reserve almost with certainty, commitment has no value. Commitment is a shorthand for a high cost to re-auction, which may reflect a concern for reputation. Several variations of the model assess the robustness of our results.

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1. Introduction

Anecdotal evidence indicates that auctioneers do not always stick to the announced rules and occasionally revoke the reserve if no bids are forthcoming. Typically, this action ignites bidding and it is not uncommon that the closing bid exceeds the initial reserve. This suggests that bidders anticipate the auctioneer's lack of perfect commitment and sometimes take a gamble and forgo meeting the reserve, hoping that the reserve is revoked and a bargain may be struck at a price below the reserve, while risking no trade.

While this behavior is endemic in low to medium caliber auctions,[1] major auction houses tend to stick to the stipulated reserve. However,

they have a similar tendency to bypass their rules by discretely re-auctioning objects that did not meet the reserve. Such behavior is reported repeatedly by those who keep track of the performance of major auction houses like Christie's and Sotheby's.[2]

In the present paper, we examine this behavior in auctions, focusing on open ascending bid (English) auctions. For this purpose we first analyze the equilibrium behavior of bidders who believe that the auctioneer revokes the reserve if it is not met with some probability. Not surprisingly, we find that bidders strategically forgo bidding if their valuation is relatively close to the initial reserve. For high valuation bidders it does not pay to take the risk of foregoing trade by not meeting the reserve.

However, bidders' beliefs cannot be arbitrary; they must be supported by rationally predicted behavior of the auctioneer. This brings us to the second ingredient of our analysis: the fact that the reserve price may inadvertently signal the auctioneer's type. Bidders have non-degenerate prior beliefs concerning the type of auctioneer, and update their beliefs after observing the reserve set by the auctioneer. If, in

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[1] A case in point is the annual Christmas Art Auction of the Berlin Senate. There, the city government auctions the work of artists who were supported by a fellowship. In these auctions, the auctioneer occasionally initiated the breach of his rules with the words "Just once, I revoke the reserve.".

[2] For example, McAfee and Vincent (1997, p. 246) report: "Regular participants in the now defunct Christies' auctions of fine wines often experience déjà vu. The same bottles or rare wine seemed to appear auction after auction." In order to understand this behavior one must keep in mind that auction houses are agents of the seller and they have to act in the interest of the seller, even though this may impair their reputation.

equilibrium, the two types of auctioneer set different reserves, bidders can perfectly infer the auctioneer's type after observing the reserve; whereas if they "pool" and set the same reserve, beliefs remain unchanged.

Like in standard signaling games where signals are pro-actively used to reveal one's type, the present game admits a multitude of equilibria that exhibit either pooling (same reserves) or full revelation (distinct reserves). This multitude is of course due to the arbitrariness of "off-equilibrium" beliefs, where Bayes' rule cannot be applied to assure consistency of beliefs with equilibrium strategies. Therefore, one needs to eliminate implausible beliefs by applying well-known equilibrium refinements such as the "intuitive criterion" introduced by Cho and Kreps (1987).[3]

After applying the intuitive criterion the number of equilibria is drastically reduced. Only those equilibria survive in which the auctioneer who is committed to never revoke his reserve price sets a reserve equal to zero. Therefore, the auctioneer who is committed not to re-auction is drastically hurt by the fact that bidders are initially (slightly) uncertain about his type to such an extent that his commitment has no value.

This finding is reminiscent of a well-known paradox exposed by Bagwell (1995) who observed in the framework of duopoly games that imperfect observability of the first mover's actions tends to destroy the value of commitment. However, in the present paper the erosion of the value of commitment is not due to imperfect observability, and it is not restricted to pure strategy equilibria, because we also cover mixed equilibria.[4]

Imperfect commitment in English auctions has also been studied by Caillaud and Mezzetti (2004) in the context of a sequence of two English clock auctions of two identical objects. They assume that the auctioneer cannot commit to a sequence of reserve prices, although he can commit to a reserve price in each of the two auctions. Subsequently, the auctioneer takes advantage of the information revealed by the outcome of the first auction to set the optimal reserve price in the second auction. In turn, bidders respond by not bidding in the first auction unless their valuation is sufficiently higher than the reserve price. Therefore, the equilibrium exhibits some pooling property.

A complementary issue has been pursued by McAfee and Vincent (1997). They consider an auctioneer who is not committed to stick to the announced reserve and re-auctions objects after some lapse of time, if the reserve has not been met, and in each auction resets the optimal reserve, based on updated information about bidders' valuations. Similar to a well-known property of the durable goods monopoly they find that the reserve converges to the seller's own valuation if the time span between auctions becomes arbitrarily small.

Another complementary issue is pursued in the auctions with resale literature, which assumes that the auctioneer cannot prevent bidders from reselling to each other, after the auction. This issue comes up if the outcome of the primary auction is inefficient, which occurs notoriously if the primary auction is a first-price auction and bidders are asymmetric (see Garratt and Troeger (2006); Hafalir and Krishna (2008); Virág (2013)).[5]

The plan of the paper is as follows: In Section 2 we spell out the assumptions. In Section 3 we analyze equilibrium bidding for given beliefs concerning the auctioneer's type, and in Section 4 we analyze the signaling equilibrium reserve prices, followed by a proof that a standard equilibrium refinement yields one unique separating and one unique pooling equilibrium (Section 5). In Section 6 we analyze three

extensions and explore what happens if the auctioneer's type is correlated with his consumption value, if the non-committed auctioneer is also subject to a small cost to re-auction, and if the committed type may also re-auction at zero cost, yet is bound to maintain the reserve set in the initial auction. The paper closes with a discussion in Section 7. Several technical proofs are in Appendices A to D.

2. Model

Consider an open, ascending-bid English clock auction, subject to a reserve price, denoted by r. There, the auctioneer continuously increases the price until there is no excess demand. Once a bidder has failed to bid at a certain price, he is no longer permitted to bid at higher prices (activity rule).

In order to keep the analysis as simple as possible, we consider the case of two bidders[6] whose valuations are *i.i.d.* random variables drawn from the differentiable c.d.f. F with support $[0, 1]$ that exhibits a positive p.d.f., f, and increasing hazard rates (log-concavity of $1 - F$).

The auctioneer is either committed to stick to the announced reserve price r (referred to as type $t = c$) or he is not committed (type $t = n$) and free to re-auction at a zero reserve price in the event when no bids are forthcoming.

Commitment is used as a shorthand for a sufficiently high cost that makes it never profitable to re-auction. That cost may reflect the loss of reputation caused by breaking the rules of the auction. In this interpretation type c may be viewed as an auctioneer who has a long-term perspective, whereas type n, for whom re-auctioning is costless, may be on the verge of exiting.[7]

Bidders have non-degenerate prior beliefs and assume that the auctioneer is not committed with probability $q_0 \in (0, 1)$. Bidders update their beliefs after observing the reserve price, applying Bayes' rule, if possible. The updated beliefs are denoted by $q(r) := Pr\{\tilde{t} = n | r\}$. We focus on symmetric equilibria.

The expected profits of the two types of auctioneer are denoted by π_c, π_n. As a benchmark, we also refer to the expected profit in the hypothetical one-shot auction, denoted by $\pi_M(r)$, and its maximizer r_M (where M is mnemonic for optimality in the sense of Myerson (1981)). By the assumed hazard rate monotonicity, r_M is unique and $\pi_M(r)$ is strictly increasing for all $r < r_M$.

3. Equilibrium bidding for given beliefs

We first solve the bidding subgames that are played after the auctioneer has announced the reserve price and bidders have updated their beliefs.

Of course, truthful bidding is a dominant strategy for those who bid. However, bidders "speculate" and strategically abstain from bidding if their valuation is greater but relatively close to the reserve price.

Proposition 1. *Bidders play a symmetric equilibrium cutoff strategy, $\gamma(r) \geq r$, and bid in the initial auction if and only if their valuation is greater or equal to $\gamma(r)$. There, $\gamma(r)$ is implicitly defined as the solution of the equation:*

$$r = (1-q)\gamma + qE(X|X \leq \gamma). \tag{1}$$

Proof. Suppose bidders play a symmetric cutoff strategy with cutoff value $\gamma \leq 1$ (in Appendix A we show that the equilibrium strategies must be cutoff strategies). Then, a bidder whose valuation is equal to the cutoff value γ must be indifferent between bidding and not bidding.

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[3] For a detailed survey of various equilibrium refinements see Van Damme (2002) and Kreps (1990, Sects. 12.7, 13.4).

[4] In Bagwell (1995), imperfect observability destroys the value of commitment in the pure strategy equilibrium of his game. However, his game has also two mixed strategy equilibria, in one of which the value of commitment is restored (see Hurkens and van Damme (1997)).

[5] An exception to this rule is Haile (2000), where resale is driven by updated information about valuations that arrives after the primary auction.

[6] In this case the English clock auction is equivalent to the Vickrey auction.

[7] In Section 6.2 we make the distinction between the two types less extreme and assume that type n is also subject to a small cost to re-auction.

If that bidder wins, he pays r; if he does not bid and no one else bids, all bidders bid truthfully if a re-auction takes place. Therefore,

$$\int_0^\gamma (\gamma - r)dF(y) = q\int_0^\gamma (\gamma - y)dF(y). \qquad (2)$$

Rearranging yields Eq. (1).

Existence and uniqueness of γ follow from the fact that the RHS of Eq. (1) is strictly increasing in γ, equal to zero at $\gamma = 0$, and equal to $\hat{r}(q) := 1 - q + qE(X)$ at $\gamma = 1$, so that for all $r \leq \hat{r}(q)$ there must be a $\gamma \leq 1$ that solves Eq. (1).

If $r > \hat{r}$, Eq. (1) has the solution $\gamma(r) > 1$ and not bidding is optimal, because even a bidder with valuation equal to 1 is better off by not bidding if his rival plays the above strategy, which follows from

$$r > \hat{r} \Leftrightarrow \int_0^1 (1 - r)dF(y) < q\int_0^1 (1 - y)dF(y).$$

Therefore, in equilibrium everyone with a valuation at or above $\gamma(r)$ bids if $r \leq \hat{r}$ and no one bids if $r > \hat{r}$.

This proves that the above cutoff strategy is the unique symmetric equilibrium.[8] $\square$

As one can easily confirm, the cutoff level γ is increasing in r and in q. If the distribution is uniform, $\gamma(r, q) = 2r/(2 - q)$.

The sellers' expected revenues are:

$$\pi_c(r, q) = 2rF(\gamma(r, q))(1 - F(\gamma(r, q))) + \int_{\gamma(r,q)}^1 \int_{\gamma(r,q)}^v zf_{(12)}(v, z)dzdv \qquad (3)$$

$$\pi_n(r, q) = \pi_c(r, q) + \int_0^{\gamma(r,q)} \int_0^v zf_{(12)}(v, z)dzdv. \qquad (4)$$

There, $f_{(12)}(v, z)$ denotes the joint p.d.f. of the two order statistics, $X_{(1:2)} > X_{(2:2)}$, of the sample of two i.i.d. random valuations.[9]

In the following Lemma we summarize some properties of the equilibrium payoffs π_n, π_c as a function of the reserve price, r, and posterior beliefs, q. These properties are used in the subsequent analysis.

Lemma 1. *One has:*

$$r > \hat{r}(q) \Rightarrow \pi_c(r, q) = 0 \quad and \quad \pi_n(r, q) = E\left(X_{(2:2)}\right) \qquad (5)$$

$$r = 0 \Rightarrow \pi_c(r, q) = \pi_n(r, q) = E\left(X_{(2:2)}\right) \qquad (6)$$

$$q = 1 \Rightarrow \pi_n(r, q) = E\left(X_{(2:2)}\right) \qquad (7)$$

$$q < 1, \ 0 < r < \hat{r}(q) \Rightarrow \pi_n(r, q) > E\left(X_{(2:2)}\right) \qquad (8)$$

$$\pi_n(r, q) \geq E\left(X_{(2:2)}\right), \quad \forall q, r. \qquad (9)$$

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[8] We mention that one may also find asymmetric equilibria. For example, if valuations are uniformly distributed, the game has an asymmetric equilibrium for $q > 2/3$, in addition to the symmetric equilibrium.

[9] $f_{(12)}(v, z) = 2f(v)f(z)$ for $v \geq z$ and zero otherwise.

Proof. Assertion (5) follows from Proposition 1.

Obviously, $r = 0 \Rightarrow \gamma = 0$; using the fact that $2(1 - F(y))f(y)$ is equal to the *p.d.f.* of the second-highest order statistic, $X_{(2:2)}$, this confirms Eq. (6).

The proof of Eq. (7) is in two steps: 1) by Eq. (1), $q = 1 \Rightarrow r = E(X|X \leq \gamma(r, 1))$. 2) Moreover, using 1):

$$2rF(\gamma(r, 1))(1 - F(\gamma(r, 1))) = 2\int_0^{\gamma(r,1)} zdF(z)(1 - F(\gamma(r, 1)))$$

$$= 2\int_0^{\gamma(r,1)} zdF(z)\int_{\gamma(r,1)}^1 dF(v)$$

$$= \int_{\gamma(r,1)}^1 \int_0^{\gamma(r,1)} zf_{(12)}(z, v)dzdv.$$

Therefore, from Eqs. (3) and (4) we get $\pi_c(r, 1) = \int_{\gamma(r,1)}^1 \int_0^v zf_{(12)}(z, v)dzdv$, and hence

$$\pi_n(r, 1) = \int_0^1 \int_0^v zf_{(12)}(z, v)dzdv = E\left(X_{(2:2)}\right).$$

In order to prove Eq. (8) suppose $r < \hat{r}(q)$. Then $\gamma(r, q) < 1$ and

$$\pi_n(r, q) - E\left(X_{(2:2)}\right) = \int_{\gamma(r,q)}^1 \int_0^{\gamma(r,q)} (r - z)f_{(12)}(v, z)dzdv$$

$$= 2rF(\gamma(r, q))(1 - F(\gamma(r, q))) - 2\int_{\gamma(r,q)}^1 \int_0^{\gamma(r,q)} zf(z)f(v)dzdv$$

$$= 2((1 - q)\gamma(r, q) + qE(X|X \leq \gamma(r, q))F(\gamma(r, q))$$

$$\times (1 - F(\gamma(r, q))) - 2E(X|X \leq \gamma(r, q))F(\gamma(r, q))(1 - F(\gamma(r, q)))$$

$$= 2(1 - q)[\gamma(r, q) - E(X|X \leq \gamma(r, q))]F(\gamma(r, q))(1 - F(\gamma(r, q))) > 0. \qquad (10)$$

The proof of Eq. (9) follows from Eqs. (5), (7) and (8). $\square$

The surprising finding is that if bidders believe that the auctioneer is type n and the auctioneer sets a positive reserve price $r < \hat{r}$, bidders with high values greater than $\gamma(r, 1)$ will bid in the initial auction. Nevertheless, the auctioneer does not benefit because his payoff is the same as if he used a reserve price equal to zero. However, type n benefits from a positive reserve $r < \hat{r}$ if bidders are unsure about his type $q < 1$.

This result is perhaps less surprising if one interprets it as an implication of the well-known revenue equivalence theorem: If the auctioneer is known to be type n, the object is sold to the bidder with the highest value, either in the initial or in the second auction; therefore, the auction with a reserve price $r \geq 0$ implements the same allocation rule as the standard one-shot auction with a reserve price equal to zero; hence, revenue equivalency applies.

4. Equilibrium reserve price(s)

The game may have equilibria in pure and in mixed strategies. The equilibria in pure strategies are either separating or pooling. The equilibria in mixed strategies are either separating, which occurs if the supports of strategies are disjoint, or (partial) pooling, which occurs if the intersection of the supports is not empty. In the following we show which of these applies

Proposition 2. *The game has separating equilibria. In each separating equilibrium type c plays the pure strategy $r_c = 0$ and type n either plays the pure strategy $r_n > 0$ or randomizes over a set of positive reserve prices. In these equilibria commitment has no value.*

Proof. In two steps: 1) we show that there is no separating equilibrium with $r_c > 0$; 2) we show that there exist separating equilibria with $r_c = 0$.

1) Suppose the game has a separating equilibrium, $(r_c, r_n, \gamma(r, q), q(r))$ with $r_c > 0$. Then, one must have $q(r_c) = 0$, $q(r_n) = 1$, and

$$
\begin{aligned}
E\left(X_{(2:2)}\right) &= \pi_n(r_n, q(r_n)) \\
&\geq \pi_n(r_c, q(r_c)) \\
&> \pi_c(r_c, q(r_c)) \\
&\geq \pi_c(0, q(0)) \\
&= E\left(X_{(2:2)}\right),
\end{aligned}
\tag{11}
$$

which is a contradiction.

This argument applies also to separating mixed equilibria, where the supports of the seller's strategies are disjoint.

2) Consider $r_c = 0$, $r_n > 0$, and the belief system: $q(r) = 0$ if $r = 0$ and $q(r) = 1$ otherwise. Then, $\pi_c(r, 1) \leq \pi_n(r, 1)$, by Eq. (4). Therefore, given the belief system $q(r)$ type c cannot benefit if he deviates from $r_c = 0$. Moreover, at the given belief system type n earns a payoff equal to $E(X_{(2 : 2)})$, for all r. Therefore, type n cannot benefit if he deviates from $r = r_n$ either.

In these equilibria commitment has no value because the auctioneer does not benefit from the fact that he is committed not to re-auction, even if bidders are only slightly uncertain about his type. □

These results have an intuitive interpretation. Suppose the game has a separating equilibrium with $r_n \neq r_c > 0$. Then, the equilibrium payoff of type n is the same as if he used a zero reserve price, by Eq. (7) in Lemma 1 whereas the equilibrium payoff of type c is equal to $\pi_M(r_c)$, which is greater than $E(X_{(2 : 2)})$. In that case type n benefits from mimicking c, because he then earns even more than type c. Therefore, this cannot be an equilibrium. The incentive to mimic type c vanishes only if type c sets a reserve price equal to zero, in which case any reserve price r_n is a best reply of type n.

Essentially, the non-existence of a separating equilibrium with $r_c > 0$ is due to the fact that type n can mimic type c and benefit from this at no cost. This part is not surprising. What is surprising is that in the separating equilibrium with $r_n \in (0, \hat{r})$, the bidders with high values submit a bid in the initial auction, yet type n does not earn a higher payoff than he would if he had set $r = 0$.

Proposition 3. *The game has a continuum of pooling equilibria in pure strategies. There, both types set the same reserve price $r_p \in [0, \bar{r}]$ for some $\bar{r} > 0$.*

Proof. Assume the dichotomous belief system

$$
q(r) = \begin{cases} q_0 & \text{if } r = r_p \\ 1 & \text{otherwise}. \end{cases}
\tag{12}
$$

Then, one obtains a pooling equilibrium in which both types set the common reserve price r_p for which

$$
\pi_c\left(r_p, q_0\right) \geq \pi_c(0, q_0) = E\left(X_{(2:2)}\right).
$$

In order to prove this, use the fact that for all $r \neq r_p$, $q(r) = 1$ and:

$$
\pi_n\left(r_p, q_0\right) \geq \pi_c\left(r_p, q_0\right) \geq \pi_c(0, q_0) = E\left(X_{(2:2)}\right) = \pi_n(r, 1) \geq \pi_c(r, 1).
$$

Therefore, neither type c nor type n can benefit by deviating from $r = r_p$.

A particular pooling equilibrium is obtained for $r_p = 0$, because obviously $\pi_n(0, q_0) = \pi_c(0, q_0)$. Another pooling equilibrium is obtained for $r_p = \bar{r}$, where $\bar{r}$ is implicitly defined as the smallest positive solution of the equation $\pi_c(\bar{r}, q_0) = E(X_{(2:2)})$.[10] □

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[10] To prove that there exists a positive $\bar{r}$ note that $\partial_r \pi_c(0, q_0) = 0$, $\partial_{rr}\pi_c(0, q_0) > 0$, and $\pi_c(1, q_0) = 0$; therefore, the equation $\pi_c(r, q_0) = E(X_{(2 : 2)})$ has a positive solution.

Lemma 2. *In a (partial) pooling equilibrium the supports of strategies have only one element in common.*

Proof. Suppose the intersection contains at least two elements, denoted by r_1, r_2. Then, both types must be indifferent between them, i.e.,

$$
\pi_c(r_1, q(r_1)) = \pi_c(r_2, q(r_2)) \quad \text{and} \quad \pi_n(r_1, q(r_1)) = \pi_n(r_2, q(r_2)).
$$

Let $q_i := q(r_i)$, $\gamma_i := \gamma(r_i, q_i)$, $i \in \{1, 2\}$. By the above one must have:

$$
\pi_n(r_1, q_1) - \pi_c(r_1, q_1) = \pi_n(r_2, q_2) - \pi_c(r_2, q_2)
$$

$$
\int_0^{\gamma_1} \int_0^v z f_{(12)}(v, z) dz dv = \int_0^{\gamma_2} \int_0^v z f_{(12)}(v, z) dz dv
$$

$$
\gamma_1 = \gamma_2 =: \gamma^*
$$

$$
0 = \pi_c(r_1, q_1) - \pi_c(r_2, q_2) = (r_1 - r_2) \int_{\gamma^*}^1 \int_0^{\gamma^*} f_{(12)}(v, z) dz dv
$$

$$
\gamma^* = 1
$$

$$
\pi_c(r_1, q_1) = 0.
$$

However, this cannot be an equilibrium, because by choosing $r = 0$ type c can assure himself an expected profit equal to $E(X_{(2 : 2)})$. □

Lemma 3. *If type n randomizes in a (partial) pooling equilibrium, the single element that is common to the supports of the strategies, which is denoted by r^*, is equal to zero.*

Proof. Suppose there is an equilibrium in which $r^* > 0$. Let $r_n \neq r^*$ be another element of the support of type n's mixed strategy.

First notice that in equilibrium one has $q(r^*) < 1$ because if $q(r^*)$ were equal to one, type c could increase his payoff by setting $r = 0$ in lieu of r^*.

In equilibrium type n must be indifferent between choosing r^* and r_n, which implies

$$
\pi_n(r^*, q(r^*)) = \pi_n(r_n, 1) = E\left(X_{(2:2)}\right).
\tag{13}
$$

By Eqs. (4) and (5) one has

$$
\pi_n(r^*, q(r^*)) = \begin{cases} E\left(X_{(2:2)}\right) & \text{if } r^* \geq \hat{r}(q(r^*)) \\ E\left(X_{(2:2)}\right) + \xi & \text{otherwise}. \end{cases}
\tag{14}
$$

Let $q^* := q(r^*)$, $\gamma^* := \gamma(r^*, q^*)$. In order to sign ξ, recall that $r^* = (1 - q^*)\gamma^* + q^* E(X|X \leq \gamma^*)$ by Eq. (1). Therefore, by an argument similar to that in Eq. (10) one obtains, for all $r^* \in (0, \hat{r})$:

$$
\xi = \pi_n(r^*, q(r^*)) - E\left(X_{(2:2)}\right) = 2(1 - q^*)(\gamma^* - E(X|X \leq \gamma^*)) F(\gamma^*)(1 - F(\gamma^*)) > 0.
\tag{15}
$$

By Eqs. (13)–(15) one must have $r^* \geq \hat{r}$. However, for all $r^* \geq \hat{r}$ one has $\gamma^* \geq 1$, and hence $\pi_c(r^*, q^*) = 0$. But then it is profitable for type c to deviate to $r = 0$ because he then earns the higher payoff $\pi_c(0, q) = E(X_{(2 : 2)})$, regardless of bidders' beliefs q. Therefore, the candidate mixed equilibrium is not an equilibrium. □

This result has a simple intuitive explanation. If there were such a partial pooling (mixed strategy) equilibrium, the payoff of type n when he chooses an action $r_n \neq r^*$ (and thus is recognized as type n with certainty) must be equal to $E(X_{(2 : 2)})$, by Eq. (7). Therefore, by the indifference condition of a mixed equilibrium, one must also have $\pi_n(r^*, q^*) = E(X_{(2 : 2)})$. But this is only possible if either $r^* = 0$ (which is excluded by assumption) or $\gamma^* \geq 1$, which implies $\pi_c(r^*, q^*) = 0$.

But then $r^* > 0$ cannot be an equilibrium strategy because by choosing $r = 0$ type c can assure himself a higher payoff equal to $E(X_{(2 : 2)})$.

Lemma 4. *In a partial pooling equilibrium, if $r^* = 0$, then type c must play the pure strategy $r_c = 0$.*

Proof. Suppose type c is choosing some $r_c > 0$ with positive probability. Then $q(r_c) = 0$ and $\pi_n(r_c, q(r_c)) > \pi_c(r_c, q(r_c)) = \pi_c(r^*, q(r^*)) = E(X_{(2 : 2)}) = \pi_n(r^*, q(r^*))$ because $r^* = 0$. That is, type n has an incentive to deviate, a contradiction. $\square$

We conclude that the game has separating equilibria; these have the property that type c plays the pure strategy $r_c = 0$. The game has also (partial) pooling equilibria; these have the property that the supports of strategies have only one element in common, r^*, and either $r^* = 0$ and type c plays the pure strategy $r_c = 0$ or $r^* > 0$ and type n plays the pure strategy $r_n = r^*$.

Commitment has value only in the one type of equilibria in which $r^* > 0$. Because in these equilibria, and only in these, type c benefits from the fact that he is committed to stick to the announced reserve. However, as we show next, these equilibria are sustained by implausible off-equilibrium beliefs that are not consistent with the intuitive criterion.

5. Equilibrium selection

The multiplicity of pooling equilibria is due to the arbitrariness of beliefs. The intuitive criterion is designed to eliminate implausible beliefs. In the present context the idea is as follows. Consider a change in beliefs from q_0 to $q = 0$. That change benefits both types because it induces truthful bidding. However, type c benefits more because the impact on the first auction is the same whereas type n is adversely affected in the second auction because the type set of the participants in the second auction is reduced from $[0, \gamma(r, q_0)]$ to $[0, r]$.

If that belief change is induced by a reduction of the reserve price, it also involves a reduction in revenue. This suggests that by choosing a sufficiently large reduction in reserve price, type c can convince bidders that he is indeed type c because he benefits whereas type n would be worse off.

Lemma 5. *All (partial) pooling equilibria with $r^* > 0$ violate the intuitive criterion. The only (partial) pooling equilibria that survive the intuitive criterion are those in which type c plays the pure strategy $r_c = 0$.*

Proof. We break down the assertion into two parts: 1) (partial) pooling equilibria with $r_c = 0$ do not violate the intuitive criterion, and 2) all other (partial) pooling equilibria violate it.

1) Suppose there exists an $r' > 0$ which is more profitable for type c if, by adopting it, he convinces bidders that he is type c, i.e., $\pi_c(r', 0) > \pi_c(0, q(0)) = E(X_{(2 : 2)})$. Then, if type n mimics type c and also plays r' and thus convinces bidders that he is type c, he benefits even more, because by Eqs. (4), (3),

$$\pi_n(r', 0) \geq \pi_c(r', 0) > E\left(X_{(2:2)}\right) = \pi_n(0, q(0)).$$

2) Consider any pooling (or partial pooling) equilibrium in which $r^* > 0$ is the unique element of the intersection of the supports of strategies. We show below that there exists an action $r' < r^*$ such that $\pi_c(r', 0) > \pi_c(r^*, q_0)$ and $\pi_n(r', 0) < \pi_n(r^*, q_0)$. Deviating to that action r' should convince bidders that the auctioneer is type c because type n cannot benefit from such a deviation even under the most favorable belief $q(r') = 0$.

First notice that $q(r^*) \geq q_0 > 0$ because type n does not randomize by Lemma 3.

Let $\Delta_i(r) := \pi_i(r, 0) - \pi_i(r^*, q(r^*)), i \in \{c, n\}$. These functions have the following properties, which are illustrated in Fig. 1:

a) $\Delta_c(0) \leq 0$, because $\pi_c(0, 0) = E(X_{(2 : 2)}) \leq \pi_c(r^*, q(r^*))$.
b) $\Delta_c(r^*) > 0$, because $\pi_c(r^*, q)$ is strictly decreasing in q.
c) From a) and b) it follows that there exists an $r_0 \in [0, r^*]$ at which $\Delta_c(r_0) = 0$.
d) Let $\gamma^* = \gamma(r^*, q(r^*))$. For all $r < r^*$ one has,

$$\pi_n(r^*, q(r^*)) = \pi_c(r^*, q(r^*)) + \int_0^{\gamma^*} \int_0^v z f_{(12)}(v, z) dz dv$$

$$\pi_n(r, 0) = \pi_c(r, 0) + \int_0^r \int_0^v z f_{(12)}(v, z) dz dv.$$

Hence, because $\gamma^* > r^*$, for $r < r^*$ one has:

$$\Delta_c(r) - \Delta_n(r) = \pi_c(r, 0) - \pi_n(r, 0) - (\pi_c(r^*, q(r^*)) - \pi_n(r^*, q(r^*)))$$
$$= \int_r^{\gamma^*} \int_0^v z f_{(12)}(v, z) dz dv > 0.$$

This already shows that type c benefits more than type n if they convince bidders that they are type c by setting an $r < r^*$.
e) By slightly increasing r above r_0 one finds an r' for which $\Delta_c(r') > 0 > \Delta_n(r')$. All reserve prices that have this property are illustrated in Fig. 1 by the interval (r_0, r_2).[11] $\square$

As one can easily confirm, the separating equilibria characterized in Proposition 2 survive the intuitive criterion. Therefore, altogether we arrive at the paradoxical conclusion:

Proposition 4. *After eliminating implausible beliefs that violate the intuitive criterion, in equilibrium type c plays the pure strategy $r_c = 0$. Therefore, commitment has no value even if the prior probability of facing an uncommitted type of auctioneer is arbitrarily small.*

6. Extensions

We now check the robustness of our results and explore what happens if[12]:

• The two types of auctioneer have different consumption values (Extension I).
• Type n has also a (small) cost to re-auction (Extension II).
• Both types can re-auction at zero cost, yet type c is committed to maintain the reserve price set in the initial auction (Extension III).

6.1. Extension I

Here, we modify the model by assuming that the two types of auctioneer have different consumption values. Specifically, we assume that the consumption value of type c, denoted by $R > 0$ is higher than that of type n; the consumption value of type n is normalized to zero.

Because type c has a positive consumption value his payoff function must account for the fact that he retains a valuable good if no sale occurs. Therefore, type c's payoff function now takes the form:

$$\Pi_c(r, q) = \pi_c(r, q) + RF(\gamma(r, q))^2, \tag{16}$$

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[11] Fig. 1 is actually an exact representation of the uniform distribution case.
[12] We are grateful to one of the anonymous referees for suggesting these extensions.
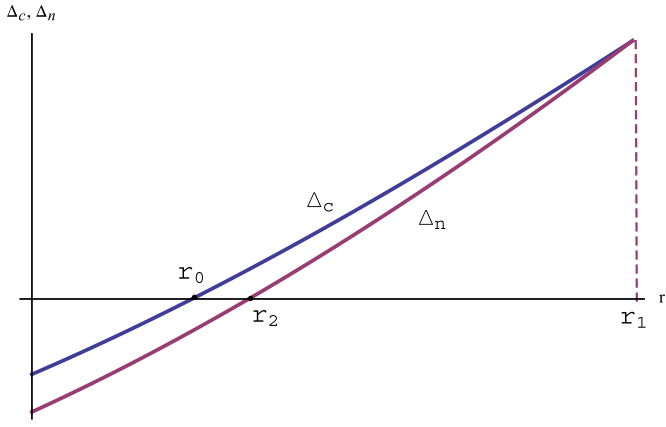
Fig. 1. Elimination of implausible beliefs by the intuitive criterion.

where π_c denotes the expected revenue, as defined in the original model.

Proposition 5. *The game has no separating equilibrium with $r_c > 0$.*

Proof. Suppose there is a separating equilibrium with strategies $r_n \neq r_c > 0$. Then $\pi_n(r_n, 1) = E(X_{(2:2)}) < \pi_n(r_c, 0)$ by Lemma 1, a contradiction. □

Proposition 6. *The strategies $r_c = r_n = r_p$ can be supported as a pooling equilibrium if and only if $\Pi_c(r_p, q_0) \geq \Pi_c(R, 1)$.*

Proof. Sufficiency: Suppose there is an r_p that satisfies the condition and assume the dichotomous belief system stated in Eq. (12). Consider type n. If he deviates to some other r, one has $q(r) = 1$ and by Lemma 1, Eq. (9):

$$\pi_n\left(r_p, q_0\right) \geq \pi_n(0, 1) = E\left(X_{(2:2)}\right) = \pi_n(r, 1).$$

Similarly, if type c deviates to some other r, one has $q(r) = 1$ and by the assumed condition and Lemma 7, which is stated and proved in Appendix B: $\Pi_c(r_p, q_0) \geq \Pi_c(R, 1) \geq \Pi_c(r, 1)$.

Necessity: Suppose there is a pooling equilibrium with r_p that violates the condition, i.e., $\Pi_c(r_p, q_0) < \Pi_c(R, 1)$. Then, type c can profitably deviate to $r_c = R$, because $\Pi_c(R, q(R)) \geq \Pi_c(R, 1) > \Pi_c(r_p, q_0)$. □

In the original model we could eliminate all pooling equilibria with $r_p > 0$ by applying the intuitive criterion. However, in this extension one cannot always eliminate all pooling equilibria with $r_p > R$. That elimination works only if R is sufficiently small. This can be explained as follows.

The elimination of pooling equilibria with $r_p > 0$ requires that one can find a deviation to $r \neq r_p$ that benefits type c but harms type n if it changes bidders' belief from q_0 to $q = 0$. If one considers the same deviations that eliminate pooling equilibria in the original model, these deviations also increase the revenue of type c and reduce the revenue of type n. However, the payoff function of type c now includes the additional term, $RF(\gamma)^2$ which upsets the elimination if R is large. This may happen because if a deviation to $r < r_p$ induces the belief change from q_0 to $q = 0$, the additional term in the payoff function of type c diminishes from $RF(\gamma(r_p, q_0))^2$ to $RF(r)^2$.

For example, if F is the uniform distribution and $q_0 = 1/2$, $R = 1/3$, $r_p = 1/2$, one finds that for all deviations, r, type n benefits more than type c.

Altogether, Extension I yields essentially the same set of equilibria if one takes into account that in the initial model the reserve price $r = 0$ can be viewed as a reserve price equal to the consumption value of type c. If R is sufficiently small, employing the intuitive criterion

makes the set of pooling equilibria shrink and, in the limit, yields the same unique pooling equilibrium.

6.2. Extension II

Here, we modify the model by assuming that type n is subject to a small cost to re-auction. We maintain the assumption that type c will never re-auction, due to a sufficiently high cost. We give a brief summary of our findings and spell out the detailed analysis in Appendix C.

Denote type n's cost to re-auction by δ. Because type n will not necessarily re-auction, we need to add the strategy σ that denotes the probability that type n re-auctions.

Also, note that bidders' belief that a second auction will take place is no longer the same as the belief that the auctioneer is type n. Denote bidders' belief that a second auction will take place by Q. Because bidder's strategy depends on Q, we reinterpret bidders' cutoff strategy γ, stated in Proposition 1, as a function of Q in place of q, i.e., $\gamma(r, Q)$. Similarly, we reinterpret the functions $\pi_n(r, q)$, $\pi_c(r, q)$ as functions of Q in place of q. This allows us to apply a number of results of the initial model to the present framework.

Consistency of bidders' beliefs with the strategy σ requires that $Q = \sigma q$ and sequential rationality requires that $\sigma(r) = 1 \Leftrightarrow E(X_{(2:2)}|X_{(1:2)} \leq \gamma(r, Q)) \geq \delta$.

The following definitions are used extensively:

$$\rho(\delta, Q) := \inf\left\{r \geq 0 | E\left(X_{(2:2)}|X_{(1:2)} \leq \gamma(r, Q)\right) \geq \delta\right\} \quad (17)$$

$$r(\delta) := \rho(\delta, 1). \quad (18)$$

Essentially, $\rho(\delta, Q)$ is the reserve price at which the auctioneer breaks even if he re-auctions, given bidders' belief Q and bidders' strategy γ. Evidently, $Q < 1 \Rightarrow \rho(\delta, Q) > r(\delta)$, and $\lim_{\delta \to 0} \rho(Q, \delta) = \lim_{\delta \to 0} r(\delta) = 0$. If F is the uniform distribution, $\rho(\delta, Q) = 3\delta(2 - Q)/2$ and hence $r(\delta) = 3\delta/2$.

Assumption. The cost parameter δ is sufficiently small so that $\rho(\delta, 0) < r_M$. In words, if type n sets r_M and bidders believe that there will be no second auction, type n will re-auction if no one bid in the initial auction.

The main results are now summarized without proofs; all proofs are in Appendix C.

Proposition 7. (Separating equilibria) *For each $r_c \in (r(\delta), \rho(\delta, 0))$ one obtains a unique separating equilibrium. There, type n re-auctions with positive probability less than one in such a way that type c is indifferent between r_c and r_n. These equilibria are consistent with the intuitive criterion. There are no other separating equilibria.*

Proposition 8. (Pooling equilibria) *For each $r_p \in [r(\delta), \rho(\delta, 0)]$ the following strategies and beliefs, together with bidders' cutoff strategy $\gamma(r, Q)$, are a pooling equilibrium:*

$$r_c = r_n = r_p, \quad \sigma(r) = \begin{cases} 0 & if \ r \leq r_p \\ 1 & otherwise \end{cases}, \quad q(r) = \begin{cases} q_0 & if \ r = r_p \\ 1 & otherwise \end{cases}. \quad (19)$$

These equilibria are consistent with the intuitive criterion. There are also other pooling equilibria; these are however not consistent with the intuitive criterion.

Note that, on the equilibrium path of pooling equilibria, no re-auction occurs.

Altogether these results indicate that:

1. Equilibrium reserve prices and the auctioneer's expected profits are bounded: $r_c, r_n \in [r(\delta), \rho(\delta, 0)]$, $\pi \in [\pi_M(r_\delta, \pi_M(\rho(\delta, 0)))]$.

2. Type n benefits and type c is harmed by bidders' uncertainty concerning the auctioneer's type.[13]
3. If δ is positive but arbitrarily small, both equilibrium reserve prices converge to zero. This holds true even if bidders believe that the auctioneer is type c almost with certainty.

The latter result proves that, as δ goes to zero, the equilibria of the perturbed game with a small cost $\delta > 0$ converge to the unique pooling equilibrium of the initial game with $r_p = 0$.

6.3. Extension III

As a last variation of our model, suppose type c may also re-auction at zero cost, yet is bound to maintain the reserve price set in the initial auction. For simplicity we explore this variation for the case of uniformly distributed values. We give a brief summary of our main findings and spell out the detailed analysis in Appendix D.

In that case, there will always be a second auction, if no one bid in the initial auction. Bidders are only uncertain whether the reserve price will be revoked or maintained.

We find that this game has no separating equilibrium. Generally, pooling equilibria cannot be supported by simple dichotomous beliefs. However, one can construct more complex beliefs that support pooling equilibria. In particular, one can find a multitude of pooling equilibria that exhibit a reserve price close to r_M (the reserve price that is optimal in the hypothetical one-shot auction).

This indicates that the results of this variation of our model are drastically different. However, this variation implicitly assumes that commitment is an innate property of the auctioneer rather than the result of the cost of running the second auction. Therefore, one should view this extension as another model rather than a robustness test of our model.

7. Discussion

Our main finding is that, after using equilibrium selection based on the intuitive criterion, the auctioneer who is committed to never re-auction, cannot benefit from a positive reserve price, even if bidders initially believe that the auctioneer is bound by his reserve almost with certainty. The intuition for this result can be sketched as follows: for type n it is less costly to set a high reserve price, because if no sale takes place, he has second chance; however, if type n sets a higher reserve price than type c, he is recognized as type n, and bidders respond in such a way that he is better off mimicking type c whenever type c sets a positive reserve; this, in turn, gives type c an incentive to distinguish himself from type n by setting a sufficiently lower reserve. This unravels until the reserve price reaches zero.

Finally, we mention that if the number of bidders is increased, the threshold level of bidder participation γ tends to diminish. In other words, the probability of bidders' speculative behavior diminishes. Therefore, the speculation induced by imperfect commitment becomes less pronounced if the number of bidders is large.

Appendix A. Supplement to the proof of Proposition 1

Here we show that bidders' equilibrium strategy must be a cutoff strategy.

Consider a measurable set $X \subset [0, 1]$. Define $m(X) := \int_X dF(x)$ and $\mu(X) := \int_X x dF(x) / m(X)$, i.e., $m(X)$ is the probability that a bidder's value belongs to X and $\mu(X)$ the average of values that belong to X.

Lemma 6. *In any equilibrium, if bidder i with value v participates in the initial auction, then all types of i with values greater than v also participate in the initial auction.*

Proof. Let X_1 be the set of the opponent's types that participate in the initial auction and X_0 be the complement of X_1. Also define

$$X_1^L := X_1 \cap \{x | x \le v\}$$

$$X_1^H := X_1 \cap \{x | x > v\}$$

$$X_0^L := X_0 \cap \{x | x < r\}$$

$$X_0^H := X_0 \cap \{x | x \ge r\}$$

$$X_0^{HL} := X_0^H \cap \{x | x \le v\}$$

$$X_0^{HH} := X_0^H \cap \{x | x > v\}.$$

Then the expected payoff of bidder i with value v if he participates in the initial auction ($U_i^P(v)$) and that if he does not participate in the initial auction ($U_i^{NP}(v)$) are

$$U_i^P(v) = m(X_0)(v-r) + m\left(X_1^L\right)\left(v - \mu\left(X_1^L\right)\right)$$

$$U_i^{NP}(v) = q\left[m\left(X_0^L\right)\left(v - \mu\left(X_0^L\right)\right) + m\left(X_0^{HL}\right)\left(v - \mu\left(X_0^{HL}\right)\right)\right].$$

For some $v' > v$ define $X_1^{HL} := X_1^H \cap \{x | x \le v'\}$ and $X_0^{HHL} := X_0^{HH} \cap \{x | x \le v'\}$. Then the expected payoff of bidder i with value v' if he participates in the initial auction ($U_i^P(v')$) and that if he does not participate in the initial auction ($U_i^{NP}(v')$) are

$$U_i^P(v') = m(X_0)(v'-r) + m\left(X_1^L\right)\left(v' - \mu\left(X_1^L\right)\right) + m\left(X_1^{HL}\right)\left(v' - \mu\left(X_1^{HL}\right)\right)$$

$$\begin{aligned} U_i^{NP}(v') = q[& m\left(X_0^L\right)\left(v' - \mu\left(X_0^L\right)\right) + m\left(X_0^{HL}\right)\left(v' - \mu\left(X_0^{HL}\right)\right) \\ & + m\left(X_0^{HHL}\right)\left(v' - \mu\left(X_0^{HHL}\right)\right)]. \end{aligned}$$

Hence, we have

$$U_i^P(v') = U_i^P(v) + \left(m(X_0) + m\left(X_1^L\right)\right)(v'-v) + m\left(X_1^{HL}\right)\left(v' - \mu\left(X_1^{HL}\right)\right)$$

$$U_i^{NP}(v') = U_i^{NP}(v) + q\left[\left(m\left(X_0^L\right) + m\left(X_0^{HL}\right)\right)(v'-v) + m\left(X_0^{HHL}\right)\left(v' - \mu\left(X_0^{HHL}\right)\right)\right].$$

Because $m(X_0) \ge m(X_0^L) + m(X_0^{HL}) + m(X_0^{HHL})$, $\mu(X_0^{HHL}) > v$, and $U_i^P(v) \ge U_i^{NP}(v)$, it follows that $U_i^P(v') > U_i^{NP}(v')$. $\square$

Corollary 1. *In any equilibrium, if bidder i with value v does not participate in the initial auction, then all types of bidder i with values less than v do not participate in the initial auction either.*

Corollary 2. *In equilibrium every bidder plays a cutoff strategy: All types of bidder i with values $v < v_i^*$ do not participate in the initial auction, and all types of i with values $v > v_i^*$ participate in the initial auction for some $v_i^* \in [r, 1]$.*

Appendix B. Supplement to Extension I

The following Lemma is used in the proof of Proposition 6:

Lemma 7.

1) $q' > q \Rightarrow \Pi_c(r, q) > \Pi_c(r, q'), \forall r \in [R, \hat{r}(q)).$

---

[13] There is one exception: in the pooling equilibrium with $r_p = r(\delta)$ type n's payoff is the same as under no uncertainty. Of course, type c's payoff is always lower under uncertainty.

2) $r' > r \geq R \Rightarrow \Pi_c(r, 1) > \Pi_c(r', 1)$.

Proof. 1) Denote $\gamma := \gamma(r, q)$ and $\gamma' := \gamma(r, q')$. The assumption that $q' > q$ implies $\gamma' > \gamma$; hence,

$$\Pi_c(r, q) - \Pi_c(r, q') = 2 \int_{\gamma'}^{1} \int_{\gamma}^{\gamma'} (z - r) f_{(12)}(v, z) dz dv$$
$$+ 2 \int_{\gamma}^{\gamma'} \int_{\gamma}^{v} (z - R) f_{(12)}(v, z) dz dv$$
$$+ 2 \int_{\gamma}^{\gamma'} \int_{0}^{\gamma} (r - R) f_{(12)}(v, z) dz dv > 0.$$

2) Denote $\gamma := \gamma(r, 1)$, $\gamma' := \gamma(r', 1)$. Then, we have

$$2 r F(\gamma)(1 - F(\gamma)) = 2 \int_{0}^{\gamma} z f(z) dz \int_{\gamma}^{1} f(v) dv.$$

Hence,

$$\Pi_c(r, 1) = 2 \int_{\gamma}^{1} \int_{\gamma}^{v} z f(z) f(v) dz dv + 2 r F(\gamma)(1 - F(\gamma)) + R F(\gamma)^2$$
$$= 2 \int_{\gamma}^{1} \int_{0}^{v} z f(z) f(v) dz dv + R F(\gamma)^2.$$

Similarly,

$$\Pi_c(r', 1) = 2 \int_{\gamma'}^{1} \int_{0}^{v} z f(z) f(v) dz dv + R F(\gamma')^2.$$

Therefore,

$$\Pi_c(r, 1) - \Pi_c(r', 1) = 2 \int_{\gamma}^{\gamma'} \int_{0}^{v} (z - R) f(z) f(v) dz dv$$
$$\geq 2 \int_{\gamma}^{\gamma'} \int_{\gamma}^{v} (z - r) f(z) f(v) dz dv$$
$$+ 2 \int_{\gamma}^{\gamma'} \int_{0}^{\gamma} (z - r) f(z) f(v) dz dv$$
$$= 2 \int_{\gamma}^{\gamma'} \int_{\gamma}^{v} (z - r) f(z) f(v) dz dv > 0.$$

The last equality holds because, by Proposition 1,

$$\int_{\gamma}^{\gamma'} \int_{0}^{\gamma} (z - r) f(z) f(v) dz dv = (F(\gamma') - F(\gamma)) \left(\int_{0}^{\gamma} z f(z) dz - r F(\gamma) \right)$$
$$= (F(\gamma') - F(\gamma)) F(\gamma) (E(X | X \leq \gamma) - r) = 0.$$

$\square$

Appendix C. Supplement to Extension II

The following Lemma prepares the proof of Proposition 7:

Lemma 8. *1) There is no equilibrium in which $r < r(\delta)$ is played with positive probability; 2) $r > \rho(\delta, 0) \Rightarrow$ it is optimal for type n to re-auction; 3) $r > r(\delta) \Rightarrow \pi_n(r, 1) = E(X_{(2:2)}) - \delta F(\gamma(r, 1))^2$.*

Proof. 1) The cutoff strategy $\gamma(r, Q)$ is increasing in r and in Q. Therefore, for $Q < 1$ one has: $\gamma(r, Q) < \gamma(r(\delta), 1)$, and thus

$$E\left(X_{(2:2)} | X_{(1:2)} \leq \gamma(r, Q)\right) < E\left(X_{(2:2)} | X_{(1:2)} \leq \gamma(r(\delta), 1)\right) = \delta.$$

If the auctioneer sets $r < r(\delta)$, by part 1) of Lemma 8 there will be no second auction; therefore, bidders bid truthfully and the auctioneer's payoff is equal to $\pi_M(r)$. Because $\pi_M(r)$ is strictly increasing at all $r < r_M$, the auctioneer can increase his payoff by setting a higher r.

2) This is true because

$$E\left(X_{(2:2)} | X_{(1:2)} \leq r\right) > E\left(X_{(2:2)} | X_{(1:2)} \leq \rho(\delta, 0)\right) = \delta.$$

3) The proof is similar to that of Lemma 1, (7), keeping in mind that Q now takes the place of q and accounting for the cost of running the second auction. $\square$

C.1. Proof of Proposition 7

In five steps: 1) we construct a candidate separating equilibrium for each r_c, 2) we show that it is an equilibrium, 3) we show that for each such $r_c \in (r(\delta), \rho(\delta, 0))$ there is no other separating equilibrium, 4) we show that the equilibrium is consistent with the intuitive criterion, and 5) we show that there is no other separating equilibrium.

1) Assume the belief system $q(r)$ and type n's continuation strategy $\sigma(r)$:

$$q(r) = \begin{cases} 0 & \text{if} \quad r = r_c \\ 1 & \text{otherwise} \end{cases}, \quad \sigma(r) = \begin{cases} 0 & \text{if} \quad r \leq r(\delta) \\ s(r_n) & \text{if} \quad r = r_n \\ 1 & \text{otherwise} \end{cases},$$

where r_n and $s(r_n)$ are constructed as follows.
Choose $r_n, s(r_n)$ in such a way so that they solve the equation system:

$$r_n = \rho(\delta, s(r_n)), \quad \pi_c(r_n, s(r_n)) = \pi_c(r_c, 0). \tag{C.1}$$

These equations have a unique solution, which we prove as follows:
a) Let γ_δ be the cutoff value for which $E(X_{(2:2)} | X_{(1:2)} \leq \gamma_\delta) = \delta$. By definition of ρ, the s that solves the equation $r = \rho(\delta, s)$ is such that (r, s) induces the cutoff strategy γ_δ for which type n is indifferent between running and not running the second auction. Because $\rho(\delta, Q)$ is continuous and strictly decreasing in Q, the equation $r = \rho(\delta, s)$ has a unique solution s for each $r \in [r(\delta), \rho(\delta, 0)]$, denoted by $s(r)$.
b) Substituting $s(r)$ into the second equation, and using the fact that each $(r, s(r))$ induces the same cutoff level γ_δ, we find:

$$\pi_c(r, s(r)) = 2 r F(\gamma_\delta)(1 - F(\gamma_\delta)) + 2 \int_{\gamma_\delta}^{1} \int_{\gamma_\delta}^{v} z f(z) f(v) dz dv.$$

That function is strictly increasing in r and has the properties: $\pi_c(r_c, s(r_c)) < \pi_c(r_c, 0)$ and $\pi_c(\rho(\delta, 0), s(\rho(\delta, 0))) = \pi_M(\rho(\delta, 0)) > \pi_M(r_c) = \pi_c(r_c, 0)$. Therefore, the second equation has a unique solution $r_n \in (r_c, \rho(\delta, 0))$.
Altogether it follows that the equation system (C.1) has a unique solution.
2) We first show that in the candidate equilibrium neither type n nor type c has an incentive to mimic the other, i.e.,

$$\pi_c(r_c, 0) = \pi_c(r_n, \sigma(r_n)) = \pi_n(r_n, \sigma(r_n)) = \pi_n(r_c, 0). \tag{C.2}$$

The first equality holds true by construction.
The second equality follows from the fact that r_n combined with the belief $Q = \sigma(r_n)$ induces the cutoff value γ_δ at which type n's expected payoff in the second auction is equal to zero.
To prove the last equality, note that if type n sets r_c, bidders believe him to be type c, bid truthfully, and hence play the cutoff strategy $\gamma = r_c$, which is below γ_δ. At γ_δ type n is indifferent between running

and not running the second auctions; therefore, at $\gamma = r_c < \gamma_\delta$ type n does not run the second auction. Hence, $\pi_n(r_c, 0) = \pi_c(r_c, 0)$. Altogether, this confirms (C.2).

Second, we show that neither type has an incentive to deviate to a reserve price $r \notin \{r_c, r_n\}$.

Suppose type n deviates to $r > r(\delta)$. Then, $\sigma = 1, q = 1 \Rightarrow Q = 1$. Therefore, by part 3) of Lemma 8, $\pi_n(r, 1) < \pi_n(0, 1) < \pi_n(r_n, \sigma(r_n))$. Similarly, type c does not benefit from a deviation to $r > r(\delta)$.

Suppose type n deviates to $r \le r(\delta)$. Then, $\sigma = 0 \Rightarrow Q = 0 \Rightarrow$ bidders bid truthfully and set $\gamma = r$ and $\pi_n(r, 0) = \pi_M(r) < \pi_M(r_c) = \pi_n(r_c, 0) = \pi_n(r_n, \sigma(r_n))$. Similarly, type c cannot benefit from such a deviation.

3) Suppose that for some $r_c \in (r(\delta), \rho(\delta, 0))$ there is another separating equilibrium strategy $r_{n'}$.

   a) $r_n' < \rho(\delta, \sigma(r_n'))$ implies that type n does not re-auction. This induces bidders to bid truthfully. Because believe that there is no second auction, regardless of whether they observe $r_{n'}$ or r_c, either type c or type n has an incentive to mimic the other types' strategy, a contradiction.

   b) $r_n' > \rho(\delta, \sigma(r_n'))$ implies $\sigma(r_n') = 1$, which leads to $\pi_n(r_n', 1) < E(X_{(2:2)})$ by part 3) of Lemma 8, a contradiction.

Hence it must be the case that $r_n' = \rho(\delta, \sigma(r_n'))$, which means that $\sigma(r_n') = s(r_n')$. Because r_n is the unique solution to the equation $\pi_c(r, s(r)) = \pi_c(r_c, 0)$, one has $\pi_c(r_n', s(r_n')) \ne \pi_c(r_c, 0)$. If $\pi_c(r_n', s(r_n')) > \pi_c(r_c, 0)$, type c has an incentive to deviate, a contradiction. If $\pi_c(r_n', s(r_n')) < \pi_c(r_c, 0)$, type n has an incentive to deviate, because $\pi_n(r_n', s(r_n')) = \pi_c(r_n', s(r_n'))$ by definition of $s(r_n')$ and $\pi_c(r_c, 0) = \pi_n(r_c, 0)$, a contradiction.

4) The above stated equilibria are consistent with the intuitive criterion, because if type c deviates and earns a higher payoff while still convincing bidders that he is type c, type n can do the same and also earn a higher payoff because the equilibrium payoffs of both types are the same (see (C.2)).

5) If the game has another equilibrium, one must have $r_c > \rho(\delta, 0)$.

Suppose there is such an equilibrium. We know that, in general, $r_n \ge r(\delta)$, and in a separating equilibrium $q(r_n) = 1$.

If $\sigma(r_n) < 1$, then there is no benefit from running the second auction (because type n is either indifferent or prefers not to run the second auction). In that case $\pi_n(r_n, \sigma(r_n)) = \pi_c(r_n, \sigma(r_n)) \le \pi_c(r_c, 0) < \pi_n(r_c, 0)$, where the last inequality follows from the fact that if type n would set $r = r_c$ he would benefit from running the second auction (by definition of $\rho(\delta, 0)$).

If $\sigma(r_n) = 1$, it follows that $Q = 1 \Rightarrow \pi_n(r_n, 1) < E(X_{(2:2)})$, by part 3) of Lemma 8.

C.2. Proof of Proposition 8

In two parts: I) the stated strategies and beliefs are pooling equilibria and they are consistent with the intuitive criterion; II) all other pooling equilibria are inconsistent with the intuitive criterion.

I) In the asserted equilibrium one has $\sigma = 0 \Rightarrow Q = 0 \Rightarrow \gamma = r_p \Rightarrow \pi_n = \pi_c = \pi_M(r_p)$.

If the auctioneer deviates and sets $r < r_p$, Q remains equal to zero because $\sigma = 0$; thus, bidders continue to bid truthfully, which implies $\gamma = r$. Therefore, $\pi_c(r, 0) = \pi_n(r, 0) = \pi_M(r) < \pi_M(r_p)$, because $\pi_M(r)$ is strictly increasing for all $r < r_M$ and $r_p < r_M$.

If the auctioneer deviates and sets $r > r_p$, one has $\sigma = 1$ and $q = 1 \Rightarrow Q = 1$. Therefore, $\gamma > r$, i.e., bidders whose values are greater than but close to r abstain from bidding in the initial auction. Hence, $\pi_c(r, 1) < \pi_n(r, 1) < \pi_n(r_p, 0) = \pi_M(r_p)$, where the second inequality follows from part 3) of Lemma 8.

Evidently, the strategy $\sigma(r)$ is sequentially rational and the stated beliefs are consistent with the stated equilibrium strategies

whenever Bayes' rule can be applied.

The above stated equilibria are consistent with the intuitive criterion, because if type c deviates, convinces bidders that he is type c, and earns a higher payoff, type n can do the same and also earn a higher payoff because the equilibrium payoffs of both types are the same, due to the fact that $\sigma(r_p) = 0$.

II) There is obviously no pooling equilibrium with $r_p < r(\delta)$. If there exists an $r_p > \rho(\delta, 0)$ for which $\pi_c(r_p, q_0) \ge \pi_c(r(\delta), 0)$, for each such r_p the following strategies and beliefs, together with bidders' cutoff strategy $\gamma(r, Q)$, are also a pooling equilibrium[14]:

$$r_c = r_n = r_p, \quad \sigma(r) = \begin{cases} 1 & \text{if } r \ge \rho(\delta, 0) \\ 0 & \text{otherwise} \end{cases}, \quad q(r) = \begin{cases} q_0 & \text{if } r = r_p \\ 1 & \text{otherwise} \end{cases}.$$
(C.3)

However, this equilibrium violates the intuitive criterion. The proof is in two steps: we show that 1) under the stated condition the constructed strategies and beliefs are an equilibrium, and 2) this equilibrium violates the intuitive criterion.

1) The auctioneer does not benefit if he deviates to $r \le r(\delta)$, by choice of r_p. If he deviates to $r > r(\delta)$, $q(r) = 1$ and type n will re-auction; therefore, $Q(r) = 1$. By part 3) of Lemma 8, $\pi_c(r, 1) \le \pi_n(r, 1) < E(X_{(2:2)})$. Therefore, neither type n nor type c benefits from such deviations.

2) Consider such a pooling equilibrium with $r_p > \rho(\delta, 0)$. Define $\Delta_c(r) := \pi_c(r, 0) - \pi_c(r_p, q_0), \Delta_n(r) := \pi_n(r, 0) - \pi_n(r_p, q_0)$, and $\gamma_p := \gamma(r_p, q_0)$, and one has,

$$\pi_n(r_p, q_0) = \pi_c(r_p, q_0) + \int_0^{\gamma_p} \int_0^v z f_{(12)}(v, z) dz dv - \delta F(\gamma_p)^2$$

$$\pi_n(r, 0) = \pi_c(r, 0) + max\left\{0, \int_0^r \int_0^v z f_{(12)}(v, z) dz dv - \delta F(r)^2\right\}.$$

Hence,

$$\Delta_c(r) - \Delta_n(r) = \pi_c(r, 0) - \pi_n(r, 0) - \left(\pi_c(r_p, q_0) - \pi_n(r_p, q_0)\right)$$

$$= \int_0^{\gamma_p} \int_0^v z f_{(12)}(v, z) dz dv - \delta F(\gamma_p)^2$$

$$- max\left\{0, \int_0^r \int_0^v z f_{(12)}(v, z) dz dv - \delta F(r)^2\right\}$$

$$= F(\gamma_p)^2 \left(E(X_{(2:2)} | X_{(1:2)} \le \gamma_p) - \delta\right)$$

$$- max\left\{0, F(r)^2 \left(E(X_{(2:2)} | X_{(1:2)} \le r) - \delta\right)\right\}.$$

Because $F(x)^2$ and $E(X_{(2:2)} | X_{(1:2)} \le x)$ are increasing in x and $\gamma_p > r$ for all $r \le r_p$, it follows that $\Delta_c(r) - \Delta_n(r) > 0$ for all $r \le r_p$. This already shows that type c benefits more than type n if they set an $r < r_p$ and convince bidders that they are type c.

Finally, we construct an $r < r_p$ for which only type c benefits if bidders are convinced that the auctioneer who sets that r is type c.

There exists an $r' \in (r(\delta), r_p)$ for which $\Delta_c(r') = 0 > \Delta_n(r')$. This is assured by the fact that $\Delta_c(r(\delta)) \le 0, \Delta_c(r_p) > 0, \Delta_c$ is continuous and increasing and $\Delta_c(r) - \Delta_n(r) > 0$ for all $r \in [r(\delta), r_p]$. Therefore, by slightly increasing r one finds the r that has the desired property. There may also exist partial pooling equilibria in which pooling occurs with positive probability at some $r > \rho(\delta, 0)$. However, if they exist, they also violate the intuitive criterion. The proof of this claim is the same as above.

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[14] The stated condition that there exists an r for which $\pi_c(r, q_0) \ge \pi_c(r(\delta), 0)$ can be satisfied. For example, if F is the uniform distribution and $\delta = 0.05$, the condition is satisfied for all $q_0 < 0.78$.
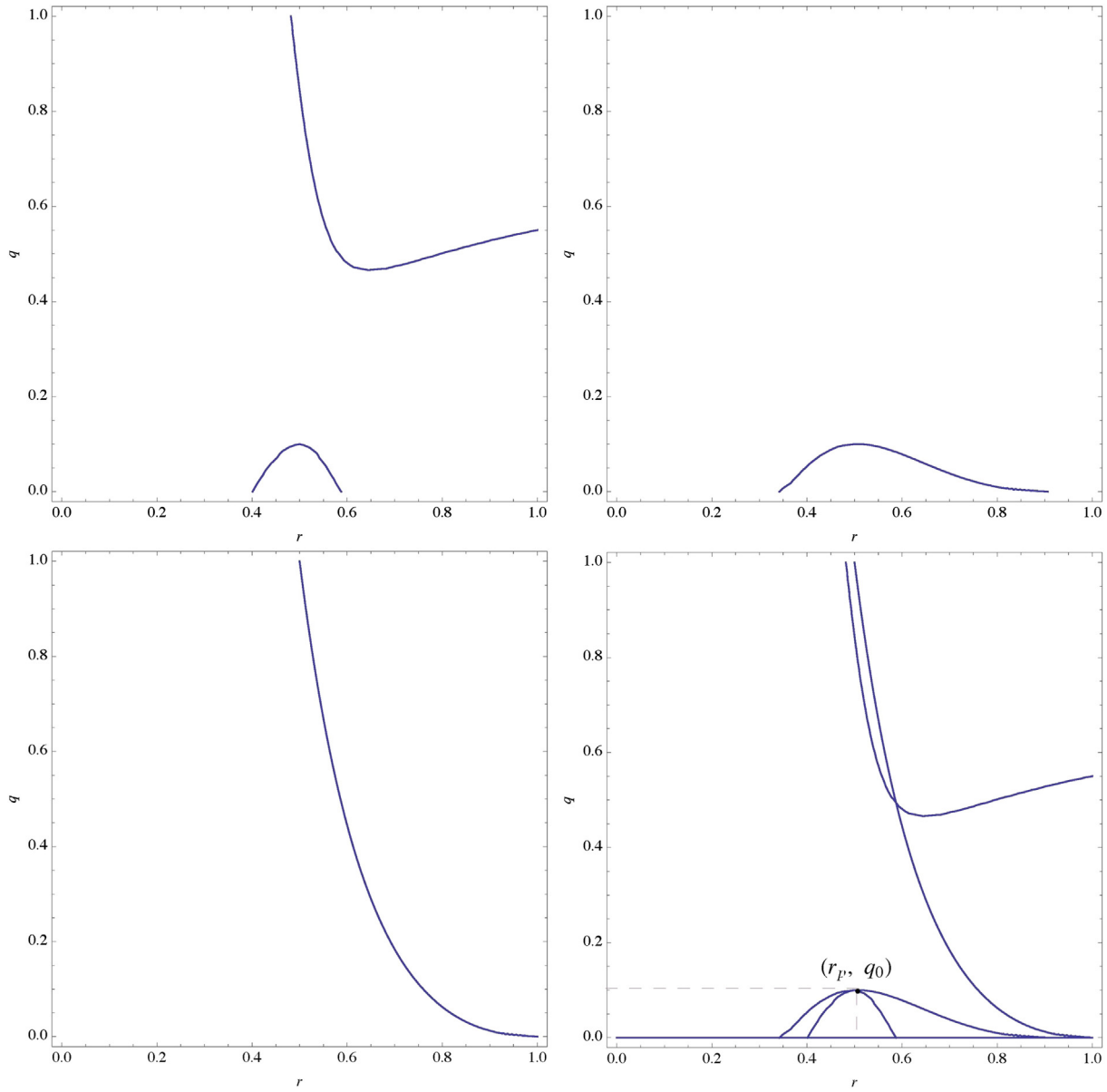
Fig. D.2. Upper left: indifference curves of type c; upper right: indifference curve of type n; lower left: area for which the indifference curves are defined; lower right: beliefs $q(r)$ that support the pooling equilibrium.

Appendix D. Supplement to Extension III

In a first step we prove that the game has no separating equilibrium.

Suppose the game has a separating equilibrium, $(r_c, r_n, \gamma(r, q), q(r))$, with $r_c > 0$. Then, one must have $q(r_c) = 0$, $q(r_n) = 1$, and

$$
\begin{aligned}
E\left(X_{(2:2)}\right) &= \pi_n(r_n, q(r_n)) \\
&\geq \pi_n(r_c, q(r_c)) \\
&> \pi_c(r_c, q(r_c)) \\
&\geq \pi_c(0, q(0)) \\
&= E\left(X_{(2:2)}\right)
\end{aligned}
$$

which is a contradiction. There, $\pi_n(r_c, q(r_c)) > \pi_c(r_c, q(r_c))$ follows from the fact that if both types set r_c they earn the same payoff in the initial auction, but type n earns additional income in the second auction. It follows that if a separating equilibrium exists, one must have $r_c = 0$.

However, $r_c = 0$ cannot be part of separating equilibrium, because type c can assure himself a minimum payoff,[15] that exceeds $E(X_{(2:2)})$.

In a second step we explain the construction of pooling equilibria, which is an elaborate exercise because simple dichotomous belief systems that penalize deviations by setting $q = 1$ cannot support pooling equilibria. We illustrate the construction for the particular prior belief $q_0 = 0.1$ and show that in this case there are pooling equilibria in which both types of auctioneer pool at a reserve price close to r_M.

The graphs in Fig. D.2 (which are exact representations of computations) illustrate the construction. There, (r_p, q_0) denotes the equilibrium point. We show that for all deviations $r \neq r_p$ there exists a belief $q(r)$ that makes that deviation unprofitable for both types of auctioneer.

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[15] To compute that minimum payoff one needs to compute the worst-case belief for each r, denoted by $\tilde{q}(r)$, and then show that $max_r \, \pi_c(r, \tilde{q}(r)) > E(X_{(2:2)})$.

The two graphs in the upper left of Fig. D.2 plot the indifference curves of type c that yield the equilibrium payoff, i.e., $\pi_c(r, q) = \pi_c(r_p, q_0)$. The area between the two graphs gives the (r, q) combinations for which type c cannot benefit by deviating from r_p.

The graph in the upper right of Fig. D.2 plots the indifference curve of type n that yields the equilibrium payoff, i.e., $\pi_n(r, q) = \pi_n(r_p, q_0)$. The area above that graph gives the (r, q) combinations for which type n cannot benefit by deviating from r_p.

In the lower left of Fig. D.2 the area to left of the graph gives the (r, q) combinations for which the indifference curves displayed in the two previous figures are defined.

In the lower right of Fig. D.2 all of the above figures are combined. By looking at the intersection of the above identified areas one can see that for each r one finds a belief $q(r)$ that makes the deviation from r_p to that r unprofitable. This shows that pooling equilibria with (r_p, q_0) do exist. Indeed, the graph of any line in the constructed area that covers all $r \in [0, 1]$ represents a belief system that supports the pooling equilibrium.

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