The implied market price of weather risk

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Weather

- Influences our daily lives and choices
- Impact on corporate revenues and earnings
- Metereological institutions: business activity is weather dependent
  - British Met Office: daily beer consumption increases by 10% if temperature decreases by 3°C
  - If temperature in Chicago is less than 0°C: consumption orange juice declines 10% in average

Examples

- Natural gas company suffers negative impact in mild winter
- Construction companies buy WD (rain period)
- Cloth retailers sell fewer clothes in hot summer
- Salmon fishery suffer losses by increase of sea temperatures
- Ice cream producers (cold summers)
- Disney World (rain period)

What are Weather derivatives (WD)?

Hedge weather related risk exposures:

- Payments based on weather-related measurements
- Underlying: temperature, rainfall, wind, snow, frost

Chicago Mercantile Exchange (CME):

- Monthly/seasonal/weekly temperature Future/Option contracts
- 24 US, 6 Canadian, 9 European and 2 Asian-Pacific cities (Tokyo & Osaka)
- From 2.2 billion USD in 2004 to 22 billion USD through September 2005

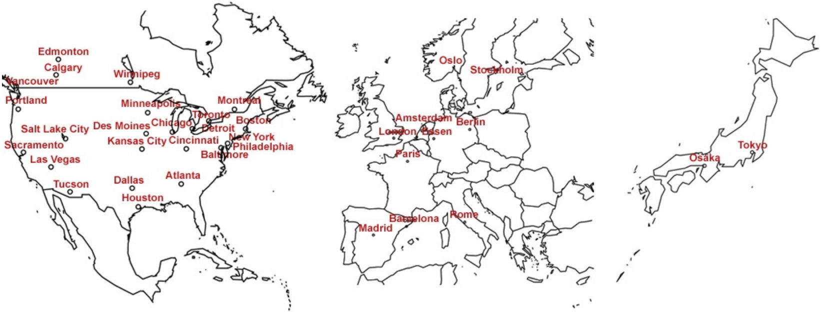


Figure 1: CME offers weather contracts on 42 cities throughout the world

Weather Derivative

Figure 2: A WD table quoting prices of May 2008 contracts. Source: Bloomberg

Weather vs. Human Capital

Investor WH organized a workshop on the 24-28 July 2006. He had a budget of £3000 for conference expenses. He estimated that for each $^\circ$C in excess of the 150°C cumulative average temperature for that week, he incurs £15 in additional costs of human capital, including water costs.

What can he do to make his plans work well?

Figure 3: Sleeping audience due to extrem hot weather

Temperature futures price

Temperature is not a tradable asset, no replication of any temperature futures: incompleteness.

An equivalent measure $Q = Q_{\theta_t}$ is pinned down to compute the arbitrage free price of a temperature future:

$$F_{(t,\tau_1,\tau_2)} = \mathsf{E}^{Q_{\theta_t}}\left[Y|\mathcal{F}_t\right] \tag{1}$$

where Y is the payoff from the temperature index and θ_t denotes the time dependent market price of risk (MPR)

By the Girsanov theorem $B_t^\theta = B_t - \int_0^t \theta_u du$ is a Brownian motion for $t \leq \tau_{\max}$,

Black-Scholes Model:

- $\theta_t = (\mu - r)/\sigma$, with the expected growth rate μ, volatility σ and the risk free interest rate r

Benth et al.(2007):

- θ_t a real valued, bounded and piecewise continous function

CME WD data

The accumulated average temperature (CAT) over $[\tau_1, \tau_2]$ days is defined as $Y = \int_{\tau_1}^{\tau_2} T_u du$, where T_t is the average temperature on day t:

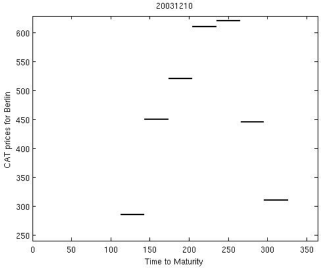


Figure 4: Berlin CAT Future Prices traded at the CME: 20031003 - 20080521. Months traded: Apr, May, Jun, Jul, Aug, Sept, Oct

Implied Market Price of Risk θ_t

CME - WD data (Bloomberg): 20031003 - 20070521

	Trading-Period		Measurement-Period	
Code	First-trade	Last-trade	τ_1	τ_2
J7	20060503	20070502	20070401	20070430
K7	20060603	20070602	20070501	20070531
M7	20060705	20070702	20070601	20070630
N7	20060803	20070802	20070701	20070731
Q7	20060906	20070904	20070801	20070831
U7	20061003	20071002	20070901	20070930
V7	20061103	20071102	20071001	20071031

Table 1: Contracts listed at the CME. J8 stands for 2008.

Implied MPR

Explicit formula for CAT futures:

$$F_{CAT(t,\tau_1,\tau_2)} = g_1(\Lambda_t) + g_2(trend_t) + g_3(\theta_t) \tag{2}$$

where Λ_t defines a seasonality function.

- Benth et al.(2007): theoretical results, θ_t=0
  - Constant volatility: underestimate prices
- Imply market price of risk θ_t from WD data:
  1. MPR θ constant or time dependent
  2. Price derivatives (future/options) and perfect replication
  3. Price non standard contract with "crazy maturities"

MPR Algorithm

Econometrics

$$
\begin{array}{c}
T_t \\
\downarrow \\
X_t = T_t - \Lambda_t \\
\downarrow \\
X_{t+3} = a^\top X_t + \sigma_t \varepsilon_t \\
\downarrow \\
\hat{\varepsilon}_t = \frac{\hat{X}_t}{\hat{\sigma}_t} \sim N(0,1)
\end{array}
$$

Fin. Mathematics.

$$
\begin{array}{c}
CAR(3) \\
\downarrow \\
F_{CAT(t,\tau_1,\tau_2)} \\
\downarrow \\
MPR
\end{array}
$$

Outline

1. Motivation ✓
2. Weather Dynamics: Berlin data
3. Stochastic Pricing Model
4. Implied Market price of risk

Berlin temperature

Seasonal function with trend: $\Lambda_t = a_0 + a_1 t + a_2 cos\left\{\frac{2\pi(t-a_3)}{365}\right\}$

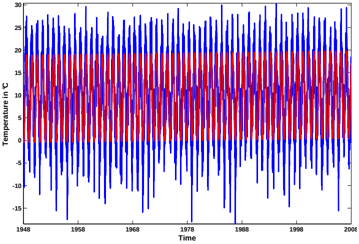


Figure 5: Seasonality effect (red line) and daily average temperatures 19480101-20080527. Source: Deutscher Wetterdienst. Seasonality estimates: $\hat{a}_0 = 8.43, \hat{a}_1 = 0.00, \hat{a}_2 = 9.79, \hat{a}_3 = -157.25$ with 95% confidence bounds, $R^2 : 0.7672$

Seasonality

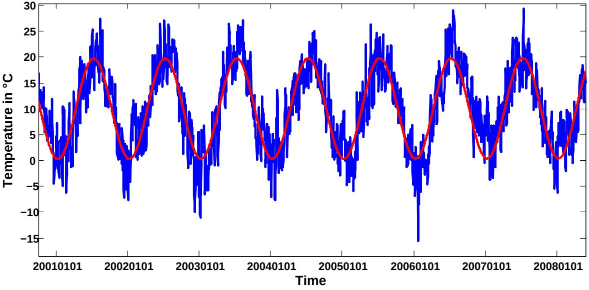


Figure 6: Daily average temperature from Berlin 20000101-20080527

Temporal dependence

Remove seasonality: $X_t = T_t - \Lambda_t$

ADF-Test:
$(1-L)X = c_1 + \mu t + \tau LX + \alpha_1(1-L)LX + \ldots \alpha_p(1-L)L^pX + \varepsilon_t$

- $\hat{\tau} = -35.001$, with 1% critical value equal to -2.5659
- Reject H_0 ($\tau = 0$), hence X_t is a stationary process $I(0)$

KPSS Test: $X_t = c + \mu t + k \sum_{i=1}^{t} \xi_i + \varepsilon_t$,

- Accept $H_0 : k = 0$ at 10% significance level that the process is stationary. The test statistic for the constant is equal to 0.653 and for the trend equal to 0.139.

PACF

AR(3): $X_{t+3} = 0.91X_{t+2} - 0.20X_{t+1} + 0.07X_t + \sigma_t \varepsilon_t$

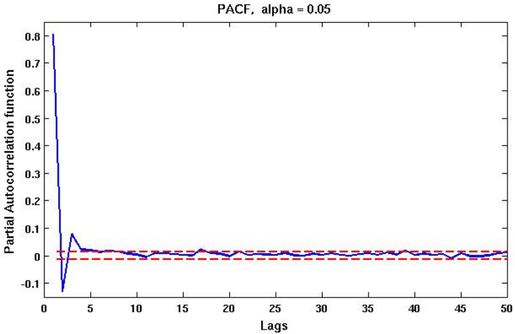


Figure 7: Partial autocorrelation function (PACF) for X_t 19480101-20080527

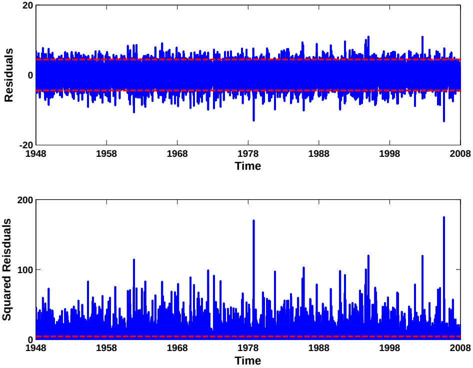


Figure 8: Residuals $\hat{\varepsilon}_t$ (up) and squared residuals $\hat{\varepsilon}_t^2$ (down) of the AR(3) during 19480101-20080527. No rejection of H_0 that the residuals are uncorrelated at 0% significance level, according to the modified Li-McLeod Portmanteau test

Seasonal volatility

Close to zero ACF for residuals and highly seasonal ACF for squared residuals of AR(3)

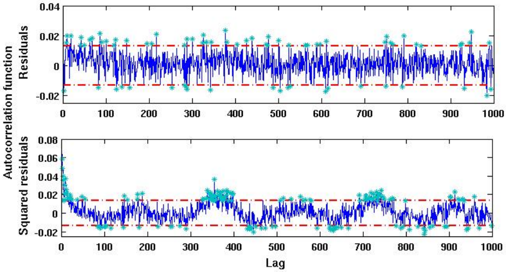


Figure 9: ACF for residuals $\hat{\varepsilon}_t$ (up) and squared residuals $\hat{\varepsilon}_t^2$ (down) of the AR(3) during 19480101-20080527

Calibration of daily variances of residuals AR(3) for 56 years:

$$\sigma_t^2 = c_1 + \sum_{i=1}^{4} \left\{ c_{2i} \cos\left(\frac{2i\pi t}{365}\right) + c_{2i+1} \sin\left(\frac{2i\pi t}{365}\right) \right\}$$

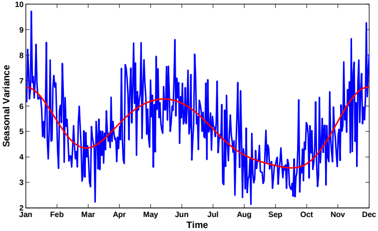


Figure 10: Seasonal variance: daily empirical variance (blue line), fitted squared volatility function (red line) at 10% significance level. $\hat{c}_1 = 5.09$, $\hat{c}_2 = 0.64$, $\hat{c}_3 = 0.74$, $\hat{c}_4 = 0.95$, $\hat{c}_5 = -0.45$, $\hat{c}_6 = 0.44$, $\hat{c}_7 = 0.05$, $\hat{c}_8 = 0.81$, $\hat{c}_9 = 0.81$

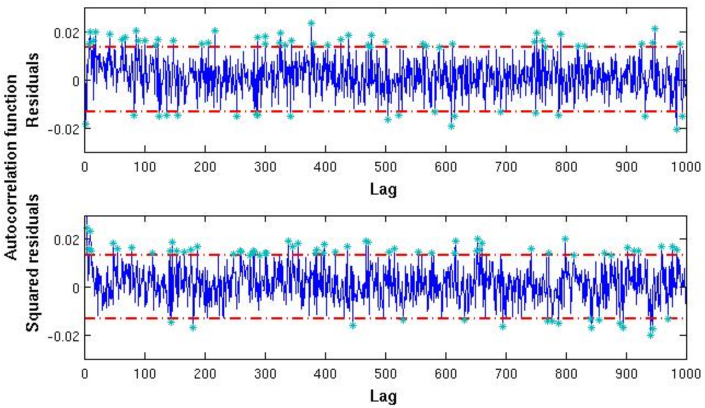


Figure 11: ACF for Berlin temperature residuals $\hat{\varepsilon}_t$ (up) and squared residuals $\hat{\varepsilon}_t^2$ (down) after correcting for seasonal volatility

Residuals become normal

Skewness= -0.08, Kurtosis= 3.56, Jarques Bera statistics equal to 319.39, acceptance of H_0: Normality, at 1% sig. level.

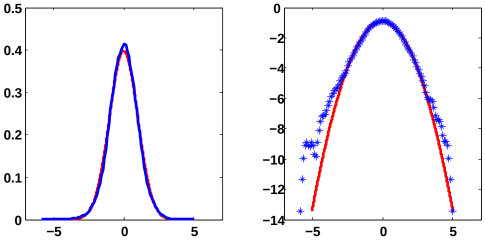


Figure 12: Left: Kernel smoothing density estimate (blue line) vs Normal Kernel (red line) for Berlin temperature residuals. Right: Log of Kernel smoothing density estimate (blue line) vs Log of Normal Kernel (gray line) for Berlin temperature residuals

Temperature dynamics

Temperature time series:

$$T_t = \Lambda_t + X_t$$

with seasonal function Λ_t. X_t can be seen as a discretization of a continuous-time process AR(p) (CAR(p)).

This stochastic model allows CAR(p) futures/options pricing.

Stochastic Pricing

Ornstein-Uhlenbeck process $\mathbf{X}_t \in \mathbb{R}^p$:

$$d\mathbf{X}_t = A\mathbf{X}_t dt + \mathbf{e}_{pt}\sigma_t dB_t$$

$\mathbf{e}_k$: k'th unit vector in $\mathbb{R}^p$ for $k = 1, ...p$, $\sigma_t > 0$, A: $p \times p$-matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \dots & & -\alpha_1 \end{pmatrix}$$

Solution of $\mathbf{X}_t = \mathbf{x} \in \mathbb{R}^p$:

$$\mathbf{X}_s = \exp\{A(s-t)\}\mathbf{x} + \int_s^t \exp\{A(s-u)\}\mathbf{e}_p\sigma_u dB_u$$

$\mathbf{X}_t$ can be written as a Continuous-time AR(p) (CAR(p)):

For $p = 1$,

$$dX_{1t} = -\alpha_1 X_{1t} dt + \sigma_t dB_t$$

For $p = 2$,

$$\begin{aligned} X_{1(t+2)} &\approx (2-\alpha_1) X_{1(t+1)} \\ &+ (\alpha_1 - \alpha_2 - 1) X_{1(t)} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

For $p = 3$,

$$\begin{aligned} X_{1(t+3)} &\approx (3-\alpha_1) X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3) X_{1(t+1)} \\ &+ (-\alpha_1 + \alpha_2 - \alpha_3 + 1) X_{1(t)} + \sigma_t (B_{t-1} - B_t) \end{aligned}$$

For Berlin temperature, consider $p = 3$:

- AR(3): $X_{t+3} = 0.91X_{t+2} - 0.20X_{t+1} + 0.07X_t + \sigma_t \varepsilon_t$
- CAR(3)-parameters: $\alpha_1 = 2.09, \alpha_2 = 1.38, \alpha_3 = 0.22$
- Stationarity condition for the CAR(3) is fulfilled:
  $\lambda_1 = -0.2069, \lambda_{2,3} = -0.9359 \pm 0.3116i$

Temperature futures price

$\exists$ Q_θ pricing so that:

$$F_{(t,\tau_1,\tau_2)} = \mathsf{E}^{Q_\theta}\left[Y|\mathcal{F}_t\right] \tag{3}$$

where Y equals the payoff and by Girsanov theorem:

$$B_t^\theta = B_t - \int_0^t \theta_u du$$

is a Brownian motion for $t \leq \tau_{\max}$. θ_t: a real valued, bounded and piecewise continous function (market price of risk)

Temperature dynamics under Q_θ

Under Q_θ:

$$d\mathbf{X}_t = (A\mathbf{X}_t + \mathbf{e}_p \sigma_t \theta_t)dt + \mathbf{e}_p \sigma_t dB_t^\theta \tag{4}$$

with explicit dynamics, for $s \geq t \geq 0$:

$$\begin{aligned} \mathbf{X}_s &= \exp\{A(s-t)\}\mathbf{x} + \int_t^s \exp\{A(s-u)\}\mathbf{e}_p \sigma_u \theta_u du \\ &\quad + \int_t^s \exp\{A(s-u)\}\mathbf{e}_p \sigma_u dB_u^\theta \end{aligned} \tag{5}$$

Temperature Indices

Heating degree day (HDD): over a period $[\tau_1, \tau_2]$

$$\int_{\tau_1}^{\tau_2} \max(c - T_u, 0) du \tag{6}$$

Cooling degree day (CDD): over a period $[\tau_1, \tau_2]$

$$\int_{\tau_1}^{\tau_2} \max(T_u - c, 0) du \tag{7}$$

c is the baseline temperature (typically 18°C or 65°F), T_u is the average temperature on day u.

Weather indices: temperature

Cumulative averages (CAT): The accumulated average temperature over $[\tau_1, \tau_2]$ days is:

$$\int_{\tau_1}^{\tau_2} T_u du \tag{8}$$

HDD-CDD parity:

$$CDD(\tau_1, \tau_2) - HDD(\tau_1, \tau_2) = CAT(\tau_1, \tau_2) - c(\tau_2 - \tau_1)$$

- Sufficient to analyse only CDD and CAT futures

CAT futures

For $0 \leq t \leq \tau_1 < \tau_2$:

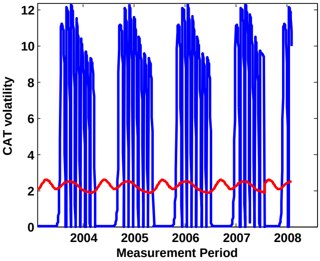
$$\begin{aligned} F_{CAT(t,\tau_1,\tau_2)} &= \mathsf{E}^{Q_\theta}\left[\int_{\tau_1}^{\tau_2} T_s ds | \mathcal{F}_t\right] \\ &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_t^{\tau_1} \theta_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p du \\ &+ \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top A^{-1}\left[\exp\{A(\tau_2 - u)\} - I_p\right]\mathbf{e}_p du \end{aligned}$$

with $\mathbf{a}_{t,\tau_1,\tau_2} = \mathbf{e}_1^\top A^{-1}\left[\exp\{A(\tau_2 - t)\} - \exp\{A(\tau_1 - t)\}\right]$,
$I_p : p \times p$ identity matrix

Benth et al. (2007)

Samuelson Effect

For contracts traded within the measurement period: CAT volatility $\sigma_t \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p$ is close to zero when the time to measurement is large. It decreases up to the end of the measurement period:



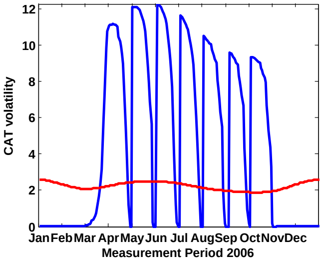


Figure 13: The Berlin CAT term structure of volatility from 2004-2008 (left side) and 2006 (right side) for contracts traded within the measurement period

Samuelson Effect

For contracts traded before the measurement period: CAT volatility $\sigma_t \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p$ is close to zero when the time to measurement is large. It increases up to the start of the measurement period:

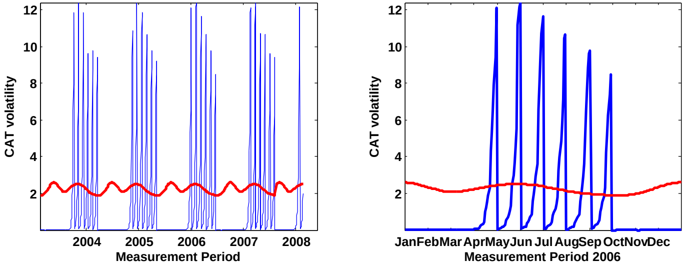


Figure 14: The Berlin CAT term structure of volatility from 2004-2008 (left side) and 2006 (right side) for contracts traded before the measurement period

Samuelson and Autoregressive effect

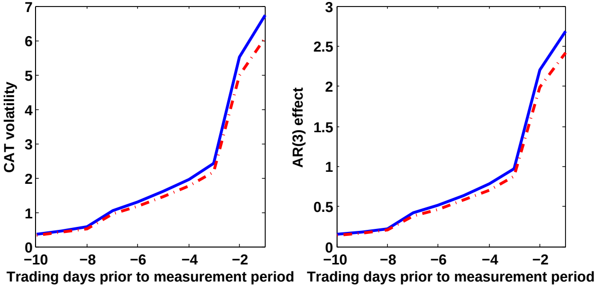


Figure 15: Berlin CAT volatility and AR(3) effect of 2 contracts issued on 20060517: one with whole June as measurement period (blue line) and the other one with only the 1st week of June (red line)

CAT call option

written on a CAT future with strike K at exercise time $\tau < \tau_1$ during the period $[\tau_1, \tau_2]$:

$$\begin{aligned} C_{CAT(t,\tau,\tau_1,\tau_2)} &= \exp\{-r(\tau - t)\} \\ &\times \Big[\left(F_{CAT(t,\tau_1,\tau_2)} - K\right)\Phi\left\{d\left(t,\tau,\tau_1,\tau_2\right)\right\} \\ &+ \int_t^\tau \Sigma^2_{CAT(s,\tau_1,\tau_2)} ds \Phi\left\{d\left(t,\tau,\tau_1,\tau_2\right)\right\}\Big] \end{aligned} \quad (9)$$

where

$$d\left(t,\tau,\tau_1,\tau_2\right) = \frac{F_{CAT(t,\tau_1,\tau_2)} - K}{\sqrt{\int_t^\tau \Sigma^2_{CAT(s,\tau_1,\tau_2)} ds}}$$

$$\Sigma_{CAT(s,\tau_1,\tau_2)} = \sigma_t \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p$$

and Φ denotes the standard normal cdf.

Hedging strategy for CAT call option

Delta of the call option:

$$\Phi\left\{d\left(t,\tau,\tau_1,\tau_2\right)\right\} = \frac{\partial C_{CAT(t,\tau,\tau_1,\tau_2)}}{\partial F_{CAT(t,\tau_1,\tau_2)}} \tag{10}$$

Hold: close to zero CAT futures when the option is far out of the money, otherwise close to 1.

CDD futures price

$$
\begin{aligned}
F_{CDD(t,\tau_1,\tau_2)} &= \mathsf{E}^{Q_\theta}\left[\int_{\tau_1}^{\tau_2} \max(T_u - c, 0)du \middle| \mathcal{F}_t\right] \\
&= \int_{\tau_1}^{\tau_2} \upsilon_{t,s}\psi\left[\frac{m_{\{t,s,\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{X}_t\}} - c}{\upsilon_{t,s}}\right] ds
\end{aligned}
$$

where $m_{\{t,s,x\}} = \Lambda_s - c + \int_t^s \sigma_u \theta_u \mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{e}_p du + x$

$$
\upsilon_{t,s}^2 = \int_t^s \sigma_u^2 \left[\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{e}_p\right]^2 du
$$

and $\psi(x) = x\Phi(x) + \varphi(x)$ with $x = \mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{X}_t$

CDD call options

$$
\begin{aligned}
C_{CDD(t,T,\tau_1,\tau_2)} &= \exp\left\{-r(\tau - t)\right\} \\
&\times \mathsf{E}\left[\max\left(\int_{\tau_1}^{\tau_2} \upsilon_{\tau,s}\psi\left(\frac{m_{\text{index}} - c}{\upsilon_{t,s}}\right) ds - K, 0\right)\right]_{\mathbf{x}=\mathbf{X}_t} \quad (11)
\end{aligned}
$$

$$
\begin{aligned}
\text{index} &= \tau, s, \mathbf{e}_1^\top \exp\left\{A(s-t)\right\} \mathbf{x} \\
&+ \int_t^\tau \mathbf{e}_1^\top \exp\left\{A(s-u)\right\} \mathbf{e}_p \sigma_u \theta_u du + \Sigma_{s,t,\tau} Y
\end{aligned}
$$

$Y \sim N(0,1)$, $\Sigma^2_{s,t,\tau} = \int_t^\tau \left[\mathbf{e}_1^\top \exp\left\{A(s-u)\right\} \mathbf{e}_p\right]^2 \sigma_u^2 du$

Infering θ_t from CME - WD data

"Implied Market Price of risk" θ_t:

1. Price of temperature derivatives
2. Price of non-standard contract with "crazy maturities"

$$
\begin{aligned}
F_{CAT(t,\tau_1,\tau_2)} &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_t^{\tau_1} \theta_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p du \\
&+ \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top A^{-1}\left[\exp\left\{A(\tau_2 - u)\right\} - I_p\right]\mathbf{e}_p du
\end{aligned}
$$

$$
\mathbf{a}_{t,\tau_1,\tau_2} = \mathbf{e}_1^\top A^{-1}\left[\exp\left\{A(\tau_2 - t)\right\} - \exp\left\{A(\tau_1 - t)\right\}\right]
$$

Constant MPR θ_t^i

$\hat{\theta}_t^i = \textit{constant}$ for each i contract, with $i = 1, 2 \ldots 7$:

$$
\begin{aligned}
\arg\min_{\hat{\theta}_t^i} \Bigg(\quad & F_{CAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathit{hat}\mathbf{X}_t \\
- \quad & \hat{\theta}_t^i \left\{ \int_t^{\tau_1^i} \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ \quad & \left. \int_{\tau_1^i}^{\tau_2^i} \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \Bigg)^2
\end{aligned}
$$

CAT Future Prices and MPR θ_t^i

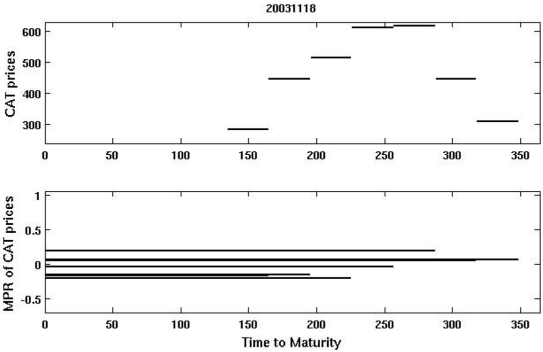


Figure 16: Berlin CAT Future Prices and MPR $\hat{\theta}_t^i$. Reject $H_0 : E(\hat{\theta}) = 0$ under the Wald statistic: 0.087 with probability 0.2322

Constant MPR θ_t

With least squares (LS), constant θ_t for all contracts at time t:

$$
\begin{aligned}
\arg\min_{\hat{\theta}_t} \Sigma_{i=1}^{7} \quad \Bigg(& \quad F_{CAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} hat\mathbf{X}_t \\
- & \quad \hat{\theta}_t \left\{ \int_t^{\tau_1^i} \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ & \quad \int_{\tau_1^i}^{\tau_2^i} \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^i - u) \right\} \right. \\
- & \quad \left. \left. I_p \right] \mathbf{e}_p du \quad \right\} \Bigg)^2
\end{aligned}
$$

LS θ_t^1, θ_t^2 constant for all contracts i at time t

Let $\hat{\theta}_t = I\left(u \leq \xi\right)\hat{\theta}_t^1 + I\left(u > \xi\right)\hat{\theta}_t^2$, with break point ξ (take e.g. half of measurement period),

$$
\begin{aligned}
\underset{\hat{\theta}_t^1, \hat{\theta}_t^2}{\arg\min}\, \Sigma_{i=1}^{7} \Bigg(\quad & F_{CAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} hat\mathbf{X}_t \\
- \quad & \hat{\theta}_t^1 \left\{ \int_t^{\tau_1^i} I\left(u \leq \xi\right) \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ \quad & \left. \int_{\tau_1^i}^{\tau_2^i} I\left(u \leq \xi\right) \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \\
- \quad & \hat{\theta}_t^2 \left\{ \int_t^{\tau_1^i} I\left(u > \xi\right) \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ \quad & \left. \int_{\tau_1^i}^{\tau_2^i} I\left(u > \xi\right) \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^i - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \Bigg)^2
\end{aligned}
$$

CAT Future Prices and MPR

$\hat{\theta}_t^1, \hat{\theta}_t^2$ constant for all contracts i at time t

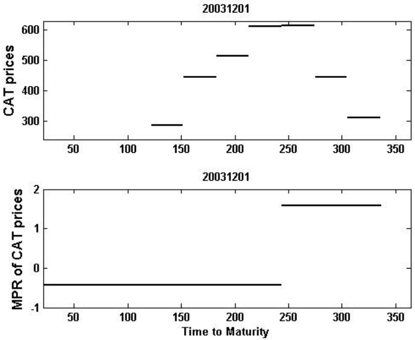


Figure 17: Berlin CAT Future Prices and MPR $\hat{\theta}_t^1$ and $\hat{\theta}_t^2$. Reject $H_0 : E(\hat{\theta}) = 0$ under the Wald statistic 0.005 with probability 0.058

MPR General case

$$
\begin{aligned}
\arg\min_{\hat{\gamma}_k} \Sigma_{i=1}^{7} \quad \Bigg(& F_{CAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_{1^i},\tau_2^i} \hat{\mathbf{X}}_t \\
- & \int_t^{\tau_1^i} \sum_{k=1}^{K} \hat{\gamma}_k \hat{h}_k(u) \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1,\tau_2} \mathbf{e}_p du \\
- & \int_{\tau_1^i}^{\tau_2^i} \sum_{k=1}^{K} \hat{\gamma}_k \hat{h}_k(u) \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^i - u) \right\} \right. \\
- & \left. I_p \right] \mathbf{e}_p du \Bigg)^2
\end{aligned}
\tag{12}
$$

where $h_k(u)$ is a vector of known basis functions, γ_k defines the coefficients.

CAT Future Prices and MPR

$h_k(u)$ obtained with splines:

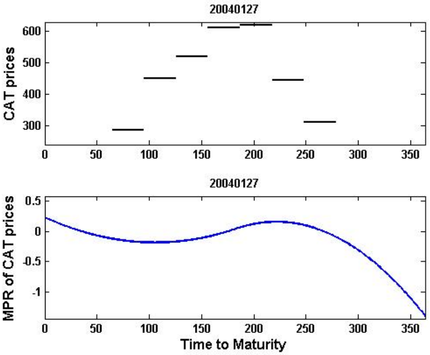


Figure 18: Berlin CAT Future Prices and MPR obtained with spline (6 knots)

MPR Bootstrap

1. $$\arg\min_{\hat{\theta}_t^1} \left(F_{CAT(t,\tau_1^1,\tau_2^1)} - \int_{\tau_1^2}^{\tau_1^2} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^1,\tau_2^1} \hat{\mathbf{X}}_t - \hat{\theta}_t^1 \left\{ \int_t^{\tau_1^1} \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^1,\tau_2^1} \mathbf{e}_p du + \int_{\tau_1^1}^{\tau_2^1} \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^1 - u) \right\} - I_p \right] \mathbf{e}_p du \right\} \right)^2$$

2. $$\arg\min_{\hat{\theta}_t^2} \left(F_{CAT(t,\tau_1^2,\tau_2^2)} - \int_{\tau_1^2}^{\tau_2^2} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^2,\tau_2^2} \hat{\mathbf{X}}_t - \int_t^{\tau_1^1} \hat{\theta}_t^1 \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^2,\tau_2^2} \mathbf{e}_p du - \int_{\tau_1^2}^{\tau_2^2} \hat{\theta}_t^2 \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^2 - u) \right\} - I_p \right] \mathbf{e}_p du \right)^2$$

3. $$\arg\min_{\hat{\theta}_t^3} \left(F_{CAT(t,\tau_1^3,\tau_2^3)} - \int_{\tau_1^3}^{\tau_2^3} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^3,\tau_2^3} \hat{\mathbf{X}}_t - \int_t^{\tau_1^1} \hat{\theta}_t^1 \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^3,\tau_2^3} \mathbf{e}_p du - \int_{\tau_1^2}^{\tau_1^3} \hat{\theta}_t^2 \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^3,\tau_2^3} \mathbf{e}_p du - \int_{\tau_1^3}^{\tau_2^3} \hat{\theta}_t^3 \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2^3 - u) \right\} - I_p \right] \mathbf{e}_p du \right)^2 \ldots$$

Smoothing the MPR:

$$\arg\min_{f\in\mathcal{F}_j} \Sigma_{t=1}^{n} \left\{\hat{\theta}_t - f(u_t)\right\}^2 = \arg\min_{\alpha_j} \Sigma_{t=1}^{n} \left\{\hat{\theta}_t - \Sigma_{j=1}^{7}\alpha_j \Psi_j(u_t)\right\}^2$$

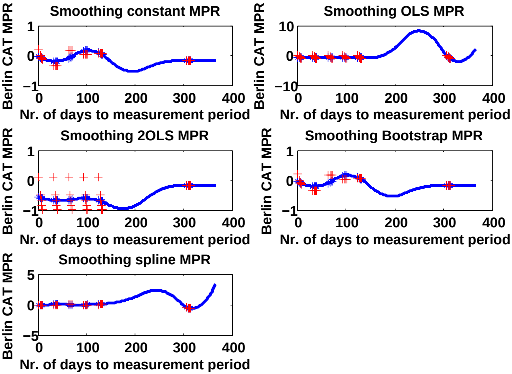


Figure 19: MPR of Berlin CAT prices for 5 lags (20060522 to 20060530) (red crosses) and smoothed MPR of Berlin CAT prices for 5 lags (blue line)

Berlin CAT Future

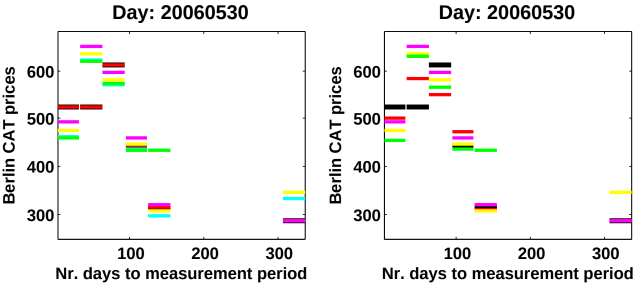


Figure 20: Right Side: Berlin CAT Future Prices from Bloomberg (black line) and estimated with constant MPR $\hat{\theta}^i_t$ (red line), MPR=0 (cyan line), constant MPR $\hat{\theta}_t$ (yellow line), 2 constant MPR $\hat{\theta}^1_t, \hat{\theta}^2_t$ (magenta line), Spline MPR (green line). Left Side: CAT Future Prices estimates using smoothed MPR's

Berlin HDD Future

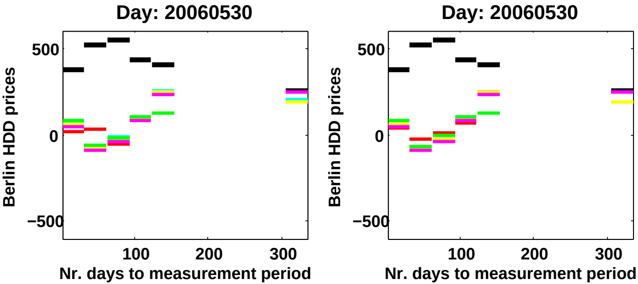


Figure 21: Right Side: Berlin HDD Future Prices from Bloomberg (black line) and estimated with constant MPR per contract per day (red line), MPR=0 (cyan line), constant MPR $\hat{\theta}_t$ (yellow line), 2 constant $\hat{\theta}_t^1, \hat{\theta}_t^2$ (magenta line), Spline MPR (green line). Left Side: HDD Future Prices estimates using smoothed MPR's

Seasonal Variation vs. MPR

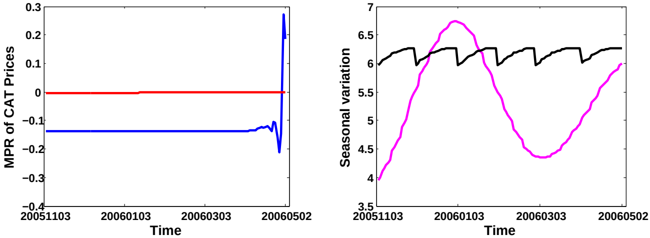


Figure 22: Left: Constant MPR $\hat{\theta}_t^i$ (blue)/smooth MPR $\hat{\theta}_t$ (red). Right: Seasonal Variation $\hat{\sigma}^2_{t+\Delta}$ (black) and $\hat{\sigma}^2_t$ (magenta) (right side) for Berlin CAT Future Prices, measurement period May 2006 (Contract K6)

Seasonal Variation $\sigma^2_{t+\Delta}$ vs. MPR $\hat{\theta}_t$

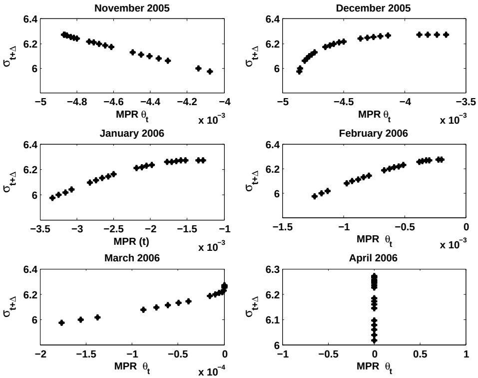


Figure 23: Seasonal Variation $\hat{\sigma}^2_{t+\Delta}$ and smoothed MPR $\hat{\theta}_t$ for Berlin CAT Future with measurement period May 2006 (Contract K6)

Seasonal Variation $\sigma^2_{t+\Delta}$ vs. MPR $\hat{\theta}^2_t$

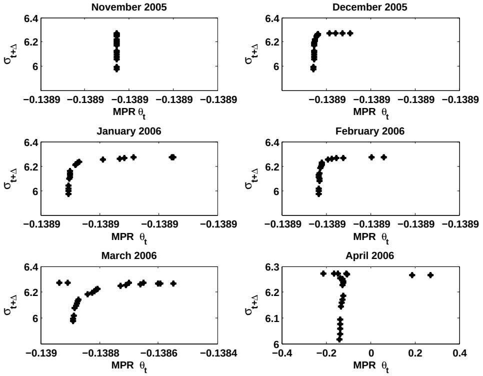


Figure 24: Seasonal Variation $\hat{\sigma}^2_{t+\Delta}$ and constant MPR $\hat{\theta}^2_t$ for Berlin CAT Future Prices, measurement period May 2006 (Contract K6)

Risk Premium (RP)

Observed

$$RP = \hat{F}_{CAT(t,\tau_1^i,\tau_2^i)} - F^P_{CAT(t,\tau_1^i,\tau_2^i)} \tag{13}$$

Implied:

$$IRP = \hat{F}_{CAT(t,\tau_1^i,\tau_2^i)} - \hat{F}_{CAT(t,\tau_1^i,\tau_2^i,\theta_t)} \tag{14}$$

$$RP = \hat{F}_{CAT(t,\tau_1^i,\tau_2^i,\theta_t)} - \hat{F}_{CAT(t,\tau_1^i,\tau_2^i,\theta_t=0)}$$

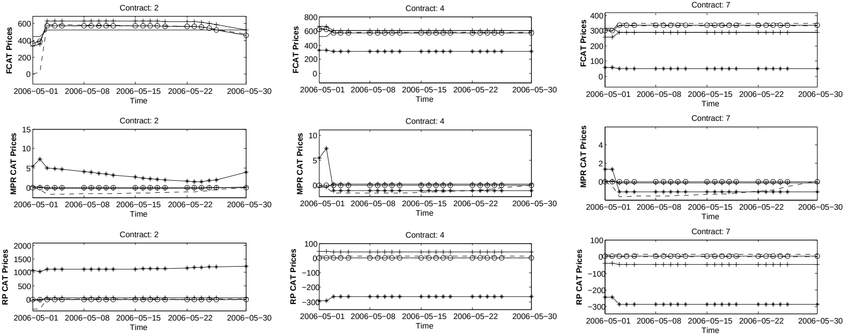


Figure 25: MPRs and RP of Berlin CAT future prices for contracts (K6, N6, H7) traded on 20060501 - 20060530: Bloomberg (black solid line), constant MPR per contract per day (: line), constant per contract day (– line), MPR=0 (-. line), two constant MPR (-* line), Bootstrap MPR (-+ line), Spline MPR (-o line).

Solution to the Human Capital problem..

Derivative Type	Call Option
Index	CAT
r	5%
t	16 December 2005
Measurement Period	24 - 28 July 2006
Strike	150°C
Tick Value	1°C= 15 £
$F_{CAT(20051216,20060724,20070728)}$	203.78
$C_{CAT(20051216,20051216,20060724,20070728)}$	53.78
$C_{CAT(20051216,20060723,20060724,20070728)}$	0

Table 2: CAT call

Outlook

- Weather Forecast: $\mathcal{F}_t$
- Volatility $\hat{\sigma}^2_{t+\Delta}$ vs. MPR $\hat{\theta}_t$
- sign of MPRs - RP: risk attitude
- Hedging

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Appendix

Residuals with and without seasonal volatility:

Lag	$Qstat_{res}$	$QSIG_{res}$	$Qstat_{res1}$	$QSIG_{res1}$
1	0.03	0.85	0.67	0.41
2	0.05	0.97	0.74	0.69
3	3.16	0.36	4.88	0.18
4	4.70	0.32	6.26	0.18
5	4.76	0.44	6.67	0.24
6	5.40	0.49	7.17	0.30
7	6.54	0.47	7.51	0.37
8	10.30	0.24	10.34	0.24
9	14.44	0.10	14.65	0.10
10	21.58	0.01	21.95	0.10

Table 3: Q test using Ljung-Box's for residuals with (res) and without seasonality in the variance (res1)

Appendix

Proof $CAR(3) \approx AR(3)$:
Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{pmatrix}$$

- Use $B_{t+1} - B_t = \varepsilon_t$
- Substitute iteratively in X_1 dynamics:

Appendix

$$
\begin{aligned}
&X_{1(t+1)} - X_{1(t)} = X_{1(t)}dt + \sigma_t\varepsilon_t\\
&X_{2(t+1)} - X_{2(t)} = X_{3(t)}dt + \sigma_t\varepsilon_t\\
&X_{3(t+1)} - X_{3(t)} = -\alpha_3 X_{1(t)}dt - \alpha_2 X_{2(t)}dt - \alpha_1 X_{3(t)}dt + \sigma_t\varepsilon_t\\
&X_{1(t+2)} - X_{1(t+1)} = X_{1(t+1)}dt + \sigma_{t+1}\varepsilon_{t+1}\\
&X_{2(t+2)} - X_{2(t+1)} = X_{3(t+1)}dt + \sigma_{t+1}\varepsilon_{t+1}\\
&X_{3(t+2)} - X_{3(t+1)} =\\
&-\alpha_3 X_{1(t+1)}dt - \alpha_2 X_{2(t+1)}dt - \alpha_1 X_{3(t+1)}dt + \sigma_{t+1}\varepsilon_{t+1}\\
&X_{1(t+3)} - X_{1(t+2)} = X_{1(t+2)}dt + \sigma_{t+2}\varepsilon_{t+2}\\
&X_{2(t+3)} - X_{2(t+2)} = X_{3(t+2)}dt + \sigma_{t+2}\varepsilon_{t+2}\\
&X_{3(t+3)} - X_{3(t+2)} =\\
&-\alpha_3 X_{1(t+2)}dt - \alpha_2 X_{2(t+2)}dt - \alpha_1 X_{3(t+2)}dt + \sigma_{t+2}\varepsilon_{t+2}
\end{aligned}
$$

Appendix

Consider 2 prob. measures P & Q. Assume that $\frac{dQ}{dP}|_{\mathcal{F}_t} = Z_t > 0$ is a positive Martingale. By *Ito's* Lemma, then:

$$\begin{aligned} Z_t &= \exp\left\{\log(Z_t)\right\} \\ &= \exp\left\{\int_0^t (Z_s)^{-1} dZ_s - \frac{1}{2}\int_0^t (Z_s)^{-2} d < Z, Z >_s\right\} \end{aligned} \quad (15)$$

Let $dZ_s = Z_s \cdot \theta_s \cdot dB_s$, then:

$$Z_t = \exp\left(\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) \quad (16)$$

Appendix

Let B_t, Z_t be Martingales under P, then by Girsanov theorem:

$$\begin{aligned} B_t^\theta &= B_t - \int_0^t (Z_s)^{-1} d < Z, B >_s \\ &= B_t - \int_0^t (Z_s)^{-1} d < \int_0^s \theta_u Z_u dB_u, B_s > \\ &= B_t - \int_0^t (Z_s)^{-1} \theta_s Z_s d < B_s, B_s > \\ &= B_t - \int_0^t \theta_s ds \qquad (17) \end{aligned}$$

is a Martingale unter Q.

Black-Scholes Model

Asset price follows:

$$dS_t = \mu S_t dt + \sigma_t S_t dB_t$$

Note that S_t is not a Martingale unter P, but it is under Q!
Explicit dynamics:

$$\begin{aligned} S_t &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma_s S_s dB_s \\ &= S_0 + \int_0^t \mu S_s ds + \int_0^t \sigma_s S_s dB_s^\theta + \int_0^t \theta_s \sigma_s S_s ds \\ &= S_0 + \int_0^t S_s(\mu + \theta_s \sigma_s) ds + \int_0^t \sigma_s S_s dB_s^\theta \qquad (18) \end{aligned}$$

Market price of Risk and Risk Premium

By the no arbitrage condition, the risk free interest rate r should be equal to the drift $\mu + \theta_s \sigma_s$, so that:

$$\theta_s = \frac{r-\mu}{\sigma_s} \tag{19}$$

In practice:
$B_t^\theta = B_t - \int_0^t \left(\frac{\mu - r}{\sigma_s}\right) ds$ is a Martingale under Q and then $e^{-rt} S_t$ is also a Martingale.

Under risk taking, the risk premium is defined as:

$$r + \Delta$$