# The Curse of Poverty and the Blessings of Wealth*

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December 2014

**Abstract**

If productivity in a society is low, then the only equilibrium involves inefficiently selfish (autarkic) behavior. If productivity is sufficiently high, then there are several equilibria that realize substantial welfare gains through reciprocal behavior. Equilibria for productive societies are distinguished by how they treat the wealth distribution. While conservative societies preserve the distribution, egalitarian behavior further improves welfare by transfers. Yet, both conservative and egalitarian (productive) societies exclude the poorest, even though that involves a welfare loss. So, there are a social and an individual poverty trap. The first is technological, the second behavioral.

* We gratefully acknowledge helpful comments and discussions by Carlos Alós-Ferrer, Egbert Dierker, and Peter Hammerstein; participants at the 2011 SAET meetings in Faro, seminar particpants at the Institute for Advanced Studies in Vienna, and the University of Konstanz; and the Economics Department at the University of the Witwatersrand in Johannesburg, South Africa, for hosting us during the course of this research. The first author acknowledges financial support from the United States National Science Foundation grant CCF-0910940 and from the Wiener Wissenschafts-, Forschungs- und Technologiefonds project, "Die Evolution von Normen and Konventionen in der Wirtschaft". The second author acknowledges financial support by the Austrian Science Fund (FWF) under project I 338-G16.

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*To those that have, shall be given.*
*Those that have not, it shall be taken from him.*

Luke 19:26

## 1 Introduction

That the social good emerges from the pursuit of private interests has been the happy message of political economy and enlightenment moral philosophy. Economists understand, however, that the alignment of private and social interests is fragile, that the invisible hand is easily made to tremble. Those who search for the social good in private action have looked in two directions: One is the hypothesis of pro-social preferences, that individual preferences inherently display some altruism to ones fellows. This project, endorsed by Adam Smith himself in the *Theory of Moral Sentiments*, is prominent today in models of strong reciprocity, inequity aversion and fairness equilibria. Second is the idea that behavior displaying regard for others is simply an expression of narrowly-defined self-interest. *Reciprocal altruism* is the idea that individuals will undertake altruistic acts on behalf of others with the expectation that they will receive altruistic beneficence in the future. It was introduced by Trivers (1971) to provide a biological explanation for altruistic behavior that benefits unrelated organisms. Axelrod and Hamilton (1981) moved reciprocal altruism to the social sphere, and Axelrod (1984) popularized the idea both within the social science community and with the public. Research into reciprocal altruism continues today, and has gained new life from recent interest in social capital. Trust, for instance, is sustained as a variety of reciprocal altruism in Abdulkadiroğlu and Bagwell (2013).

The social good becomes a private desire in models of reciprocal altruism because "what goes around comes around": Bad behavior has bad future consequences for patient players. The modeling strategy for developing this idea usually employs an infinitely repeated game, and makes use of the folk theorem or arguments from the folk theorist's toolkit, to claim that cooperation emerges as a Nash or subgame-perfect or perfect-Bayesian equilibrium of the game. While some regard this analysis as explanatory (Kandori 1992, Deb 2011), others remain unconvinced (Bowles and Ginitis 2005, Gintis, Bowles, Boyd, and Fehr 2003). This disagreement mirrors a more abstract concern about the meaning of the folk theorems. Some game theorists treat the folk theorems as positive results—repeated games with patient players are just another example of a class of games with many equilibria—others argue that the folk theorems expose the limits of game theory by identifying a class of games for which game theory has little to say so far.

No matter what side one takes in the debate on the meaning of the folk theorems in infinitely repeated games, one must concede that the repeated game model is very bare bones. The strategic environment does not vary over time, and so the only intertemporal link comes from the evolution of beliefs. While this is an excellent model for uncovering fundamental properties of games, its application to society's moral economy is dubious. Our purpose in this paper is to enrich the model in a minimal way, to make it more representative of social situations in at least one dimension.

Our model is motivated by Gerald Wilkinson's (1984, 1990) research on vampire bats. As summarized in *Scientific American*, "Two nights without a blood meal and a vampire bat starves to death—unless it can solicit food from a roostmate. A buddy system ensures that food distribution among the bats is equitable." This phenomenon cannot be modeled as a conventional repeated game, since the blood stock of the other bats and the length of time since feeding may determine any given bat's willingness to share. Whether or not Wilkinson has uncovered non-human reciprocity—and this is contested (Hammerstein 2003)—it suggests that stochastic games admitting state variables other than beliefs may provide more suitable models for investigating the evolution of cooperation.

Our interest is in humans rather than bats, so we are not wedded to the particular details of bat roosts. (However, Wilkinson reports that sharing is primarily among females, so we nod to biology by referring to our players with feminine pronouns.) We describe a problem which requires the stochastic production of a physical commodity and the maintenance of an inventory stock. Individuals consume from the stock, and an individual who cannot consume disappears. We ask, will individuals faced with production risk choose to cooperate in order to share risk? It is no surprise that our answer depends upon individuals' patience. The importance of the discount factor for supporting cooperation can be seen both theoretically (which is obvious) and empirically (Gächter, Königstein, and Kessler 2004). Impatient individuals have no desire to cooperate in order to ensure good outcomes in the future. More important, however, our answer depends upon the nature of the production process. A pair of sufficiently unproductive individuals will not be able to cooperate. Equilibria will be inefficient and lives will be short. Cooperation is only sustainable in sufficiently productive pairs, and the amount of sharing that can be sustained in equilibrium depends upon the expected output of production. In no case, however is the first-best sharing rule ever an equilibrium.

Section 2 describes the two version of the model and the specific strategies that we employ. Section 3 established bounds on the welfare that can be had in the model. Section 4 studies equilibria for both versions of the model, and Section 5 concludes. All proofs are relegated to the appendix.

## 2 The Model

Two individuals $i = 1, 2$ play an infinite-horizon stochastic game (Shapley 1953) with complete information. Individuals begin with initial inventory stocks $\mathbf{b} = (b_{10}, b_{20})$. In each period a production process delivers a random amout of the single consumption good to each individual. The delivered output is independent across individuals and time. Each delivery is two units with probability $p \in (0, 1)$ and zero with the complementary probability. Each individual maintains a stock of the consumption good. This inventory does not depreciate. We say that the individual "succeeds" if two units of consumption good are delivered; otherwise she "fails".

Each period $t$ begins for individual $i$ with the date-$t$ realization of her stochastic production process. If she succeeds, her stock increments by two units; if it fails, it remains unchanged. Next transfers are made, from one individual to another, and each individual $i$'s stock is incremented by the net transfer. Finally, if $i$ has any inventory she eats one unit (and her stock is decremented by one). If she has no inventory she dies. Thus if her initial stock was non-negative, her end-of-period stock is her initial stock plus the realization of the production process plus the net transfer less one. Each individual maximizes the expected discounted sum of the days of her life. She is alive so long as her end-of-day stock is non-negative. Finally, individual $i$ can condition her transfer on the outcome of both individuals' production, current stocks, and the entire past history of interaction, subject to a feasibility constraint we specify below.

So long as two individuals are alive, the aggregate stock can change only by $\pm 2$ or 0; either both produce or one produces and one does not or neither produce, and then two units of the consumption good are eaten. Thus there are two invariant lattices (for two individuals) in the inventory space, corresponding to even or odd parity for the aggregate stock. For simplicity we will suppose that $b_{10} + b_{20}$ is even, which constrains us to the even integer lattice when each individual's stock is positive. When only one individual is alive, the aggregate stock will move up or down by 1 unit. An inventory state $\mathbf{b} = (b_1, b_2)$ represents $b_1$ units of stock for individual 1 and $b_2$ units of stock for individual 2. The set of *inventory states* is thus

$$\begin{aligned}\mathcal{B} = \{(b_1, b_2) \in \mathbb{N}_+^2 \big| b_1 + b_2 = 2k,\ k \in \mathbb{N}_{++}\} \\ \cup \{(-1, n) \big| n \in \mathbb{N}_+\} \cup \{(n, -1) \big| n \in \mathbb{N}_+\} \cup \{(-1, -1)\}\end{aligned}$$

where $b_i = -1$ represents death.

A *social state* is a quadruple $s = (b_1, b_2, \omega_1, \omega_2)$ which describes current inventory levels and the current-period outcome of the production process; $S$ denotes the set of all states. A length-$t$ history is a sequence $h_t = (s_0, \dots, s_t)$ of game states. A history $h_t$ is a record of all production deliveries and inventory levels through date $t$. Transfers at date $t$ can depend not just on the current social state but also on the entire history of the society through date $t$.

We describe two games for each social environment, distinguished by the their feasible transfers: The *flow game* $\mathbf{G_o}$ allows transfers across individuals only from the current-period output of the production process. We suppose that the output is either transferred or irreversibly transformed to individual-specific capital. In the *stock game* $\mathbf{G_s}$, however, an individual may transfer any (integer) amount of her stock after the output is realized, constrained only by the requirement that she leave herself one unit to consume in that period. A *strategy* for individual $i$ in either game is a function which describes a transfer for each possible history of the game. These games are described informally in the remainder of this section, and the subsequent sections contain intuition for the results. Formal proofs of the theorems are in the appendix.

Although a particular transfer strategy may be history dependent in a complex way, the set of feasible transfers at any history $h_t$ are constrained only by the current game state. In the flow game $\mathbf{G_o}$, an individual can transfer only out of current period output net of consumption: nothing if the production outcome is 0, and either 0 or 1 unit if the production output is 2. In the stock game $\mathbf{G_s}$, an individual can transfer any stock she has after consumption, that is, any amount between 0 and $b_{it} + \omega_{it} - 1$. Denote by $\sigma_{it}(h_t)$ the amount individual $i$ transfers at history $h_t$, by $\sigma_i$ the entire sequence of individual $i$'s transfer functions, and by $\sigma$ the pair $(\sigma_1, \sigma_2)$.[1] Individual $i$'s stock evolves as:

$$b_{i\,t+1} = b_{it} + \omega_{it} - 1 - \sigma_{it}(h_t) + \sigma_{3-i\,t}(h_t);$$

tomorrow's beginning-of-period stock for individual $i$ is today's stock plus the outcome of her production process less what she eats less what she shares plus what the other individual shares. Dead individuals cannot be resurrected; if $b_{it} = -1$, then $b_{i\tau} = -1$ for all $\tau > t$, and of course only the living can share.

An individual's payoff is the length of her life, suitably discounted. The date of an individual's death is determined by how individuals share, that is, $\sigma$, the initial inventory stocks $\mathbf{b} = (b_{10}, b_{20})$, and the realization $\omega$ of outputs. Denote by $\tau_i(\omega; \sigma, \mathbf{b})$ the date at which individual $i$ dies (which is often abbreviated to $\tau_i$),

$$\tau_i(\omega; \sigma, \mathbf{b}) = \begin{cases} \inf\{t \in \mathbb{N}_+ \,|b_{it} = -1\}, & \text{if the set is not empty,} \\ +\infty & \text{otherwise.} \end{cases}$$

Individual $i$'s payoff function is the discounted sum of the days of her life,

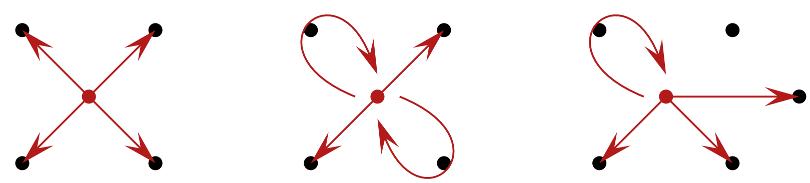
$$u_i(\omega, \sigma, \mathbf{b}) = \sum_{t=0}^{\tau_i(\omega;\sigma,\mathbf{b})-1} \delta^t = \frac{1}{1-\delta}\left(1 - \delta^{\tau_i(\omega;\sigma,\mathbf{b})}\right). \tag{1}$$

Her payoff thus depends upon the sequence $\omega$ of realizations of the production process, the initial inventory stocks, and the strategy profile of the two individuals. Individuals have identical discount factors $\delta \in (0, 1)$.

Strategies in this game are complicated objects, and in principle there can be many Nash equilibria. We have some things to say about the set of Nash equilibria, and we also examine some particular Nash equilibria which make use of some intuitive strategies. These intuitive strategies choose transfers which depend only upon the current social state, ignoring the rest of history. They are best described using *rules*. A *rule* maps game states into transfers. Strategies which depend only upon the current social state assign a rule to each inventory state. Each rule in turn associates a transfer to each production outcome.

In the game $\mathbf{G_o}$ we study three simple rules, which are illustrated in Figure 1. In the Figure, vertices represent points in the interior of the even-parity sublattice of the interior of $\mathcal{B}$. Arrows represent allowable transitions. It is

[1] The function $\sigma_i$ is actually a strategy for player $i$ in the stochastic game.



(a) Autarky (b) Simple Sharing (c) Wealth-Based Sharing

Figure 1: Rules of transition

assumed that the individual 2, whose stock is measured on the vertical axis, has a greater stock than does individual 1.

*Autarky*: Neither individual shares. Hunting adds to and eating subtracts from each individual's own stock. This is the only feasible rule when only one individual is alive.
*Simple Sharing*: An individual transfers one unit if and only if she was successful and the other failed. The simple sharing outcome depends only upon the outcome of the production process and not upon stocks.
*Wealth-Based Sharing*: The wealthier individual shares with the poorer individual whenever the wealthier individual is successful, regardless of the realization of the poorer individual's production process.
We consider one additional rule in the game $\mathbf{G_s}$.
*Full Sharing*: The wealthier individual transfers enough to the poorer individual that, at the end of the period, each holds half the total (social) inventory.

# 3 Welfare Bounds

We begin by establishing welfare benchmarks. What is the worst that can happen to rational individuals, and what is the best society could hope to do? The solution to these problems are the *autarkic value function* and the *social welfare function*, respectively.

### 3.1 Autarky

The first step is to ask what an individual can do without any social assistance. This subsection will also serve to introduce the second-order difference equations that we exploit throughout the paper.

The value $v^A$ for individual $i$ depends only on her own inventory stock $b$. It can be described recursively: The value of having inventory stock $b > 0$ in autarky is the value of being alive today plus the discounted expected value of tomorrow's stock. In inventory state $b = 0$, failure leads to death, and so the value of autarky is the expected value of being alive today plus the discounted expectation of the future welfare of succeeding production today. Formally, the value function for autarky is defined by a second-order difference equation:

$$\begin{aligned} v^A(0) &= p + \delta p v^A(1), \\ v^A(b-1) &= 1 + \delta p v^A(b) + \delta(1-p) v^A(b-2), \text{ for all } b > 0, \end{aligned} \tag{2}$$

where $b$ refers to the stock of individual $i$. This second-order difference equation has two boundary conditions. One is given by $v^A(-1) = 0$ (from which we derive the first equation in (2)). The second is given by the condition that the value of any state is bounded. An upper bound is the value of surely living forever, which is $1/(1-\delta)$.

The characteristic polynomial has two real roots, but the boundedness condition damps out the larger root, and so the unique bounded solution to this linear second-order difference equation is

$$v^A(b) = \frac{1}{1-\delta}\left(1 - \frac{\mu^{b+1}}{\delta}\right), \text{ for all } b \geq 0, \text{ where} \tag{3}$$

$$\mu = \frac{1 - \sqrt{1 - 4\delta^2 p\,(1-p)}}{2\delta p}. \tag{4}$$

This can be verified by direct substitution. As the discount factor $\delta$ approaches 1, the limit of $\mu$ is given by

$$\lim_{\delta \nearrow 1} \mu = \begin{cases} 1 & \text{if } p < 1/2, \\ \dfrac{1-p}{p} & \text{if } p \geq 1/2. \end{cases} \tag{5}$$

and, accordingly, the limiting value function in the case $p < 1/2$ and for $b \geq 0$

is

$$\lim_{\delta \nearrow 1} v^A(b) = \frac{b+2p}{1-2p}, \tag{6}$$

which is the expected value of the length of a life given initial stock $b$. When $p > 1/2$, the probability of eternal survival is no longer negligible because the production process is a random walk with positive drift. Thus for the autarkic strategy $\sigma^A$ with death date $\tau^A$,

$$\begin{aligned} v^A(b) &= \frac{1}{1-\delta}\left(1 - \Pr\left\{\tau^A < \infty\right\} E\left\{\delta^{\tau^A} \middle| \tau^A < \infty\right\}\right) \\ &> \frac{1}{1-\delta}\left(1 - \Pr\left\{\tau^A < \infty\right\}\right). \end{aligned}$$

For $p > 1/2$ this will diverge as $\delta$ converges to 1, because the probability of surviving forever is positive. So, for this case we will utilize the average discounted value, whose undiscounted limit is

$$\lim_{\delta \nearrow 1}(1-\delta)v^A(b) = 1 - \left(\frac{1-p}{p}\right)^{b+1}.$$

## 3.2 The Planner's Problem

We take as a welfare function the sum of the discounted lifetimes of the two individuals. The value of maximal social welfare will depend, of course, on which strategies we allow. Strategies available in the output game $\mathbf{G_o}$ are a subset of those available in the stock game $\mathbf{G_s}$, so an upper bound on feasible social welfare in both games is given by constructing the social welfare function using stock game transfers. However, since for each game we will be interested in how well equilibrium can do relative to the best that can be achieved with the given constraints, we will want both social welfare functions.

### 3.2.1 The Flow Game $\mathbf{G_o}$

The welfare function, and every value function we discuss below, is described recursively on the state space by means of a Bellman equation. For $(b_1, b_2) \gg 0$,

$$\begin{aligned} TW(b_1, b_2) = 2 &+ \delta(1-p)^2 W(b_1 - 1, b_2 - 1) \\ &+ \delta p(1-p)\max\{W(b_1+1, b_2-1), W(b_1, b_2)\} \\ &+ \delta(1-p)p\max\{W(b_1-1, b_2+1), W(b_1, b_2)\} \\ &+ \delta p^2 \max\{W(b_1+2, b_2), W(b_1+1, b_2+1), W(b_1, b_2+2)\}. \end{aligned} \tag{7}$$

The welfare in a given state $(b_1, b_2) \gg 0$ is the sum of the two additional periods of life that results from both players having positive inventories, plus the discounted expected value of deploying the optimal plan in the future. The four future cases correspond to both individuals failing to produce, the two cases of one producing positive output and the other not, and both producing output. The contingencies at the boundary of the state space and at the origin are different. The Bellman equation for these remaining cases, for inventory states $(0, b)$, states $(b, 0)$, and state $(0, 0)$, is described as follows: For $(b_1, b_2) = (0, b)$ with $b > 0$ (and similarly for $(b_1, b_2) = (b, 0)$) the equation

$$\begin{aligned} TW(0,b) &= 2 - (1-p)^2 + \delta (1-p)^2 v^A (b-1) \\ &+\delta p^2 \max\{W(1,b+1), W(2,b), W(0,b+2)\} \\ &+\delta p(1-p) \max\{W(1,b-1), W(0,b)\} \\ &+\delta p(1-p) \max\left\{W(0,b), v^A(b+1) - 1/\delta\right\} \end{aligned} \tag{8}$$

holds, and for $b_1 = b_2 = 0$ it holds that

$$\begin{aligned} TW(0,0) = 2\left(1-(1-p)^2\right) + \delta p^2 \max\{W(1,1), W(2,0), W(0,2)\} \\ +2\delta p(1-p) \max\left\{W(0,0), v^A(1) - 1/\delta\right\}. \end{aligned} \tag{9}$$

Uniqueness of the solution to the Bellman equation $W = TW$ is guaranteed by a contraction mapping argument, which can be found in the online Appendix A. We denote by $W^o_\delta$ the welfare function for the output game. Some of its properties can be inferred directly from the Bellman equation, but obtaining an explicit solution does not appear to be a tractable problem. Nonetheless, we have the following description of socially optimal behavior.

**Proposition 1.** *The policy of wealth-based sharing off the diagonal of $\mathcal{B}$, and simple sharing on the diagonal, is socially optimal in the flow game $\mathbf{G_o}$.*

The proposition states that the distribution of wealth has welfare consequences. The more evenly inventories are distributed, the better off is the society as a whole. Intuitively a more balanced distribution acts as an insurance device against adverse events.

Although we cannot compute the full social welfare function, simple sharing leaves the diagonal invariant. This invariance makes it possible to compute the social welfare function on the diagonal,

$$W_\delta^o(b,b) = \frac{2}{1-\delta}\left(1 - \frac{\lambda^{b+1}}{\delta}\right) \text{ for all } b \geq 0, \text{ where} \tag{10}$$

$$\lambda = \frac{1 - 2\delta p(1-p) - \sqrt{1 - 4\delta p(1-p)}}{2\delta p^2}. \tag{11}$$

For future reference observe that as $\delta$ goes to 1 the root $\lambda$ approaches

$$\lim_{\delta \nearrow 1} \lambda = \begin{cases} 1 & \text{if } p \leq 1/2, \\ \dfrac{(1-p)^2}{p^2} & \text{if } p > 1/2. \end{cases} \tag{12}$$

Therefore, for $p \leq 1/2$ and $b \geq 0$, the limiting welfare function along the diagonal of $\mathcal{B}$ is given by

$$\lim_{\delta \nearrow 1} W_\delta^o(b,b) = \frac{2b + 4p}{1 - 2p}, \tag{13}$$

which coincides with twice the limiting value of autarky found in equation (6). When $p \geq 1/2$, welfare is unbounded as $\delta$ converges to 1 because the probability of surviving forever is positive. In this case we look at the average discounted welfare. For $b \geq 0$, average discounted welfare is

$$\lim_{\delta \nearrow 1} (1-\delta) W_\delta^o(b,b) = 2\left(1 - \left(\frac{1-p}{p}\right)^{2b+2}\right). \tag{14}$$

#### 3.2.2 The Stock Game $\mathbf{G_s}$

The stock game allows unlimited transfers of inventories, subject only to the constraint that post-transfer invetories are non-negative. The Bellman equation for $(b_1, b_2) \gg 0$ is:

$$\begin{aligned} TW(b_1,b_2) = 2 &+ \delta(1-p)^2 \max_{0 \leq z \leq b_1+b_2-2} W(b_1+b_2-2-z, z) \\ &+ 2\delta p(1-p) \max_{0 \leq z \leq b_1+b_2} W(b_1+b_2-z, z) \\ &+ \delta p^2 \max_{0 \leq z \leq b_1+b_2+2} W(b_1+b_2+2-z, z), \end{aligned} \tag{15}$$

for $(b_1, b_2) = (0, b)$ with $b > 0$ (and similarly for $(b_1, b_2) = (b, 0)$) it is

$$TW(0,b) = 2p^2 + \delta p^2 \max_{0 \le z \le b+2} W(b+2-z, z)$$
$$+2p(1-p) + 2p(1-p) \max \left\{ 1 + \delta \max_{0 \le z \le b} W(b-z, z), \delta v^A(b+1) \right\} \tag{16}$$
$$+(1-p)^2 + (1-p)^2 \max \left\{ 1 + \delta \max_{0 \le z \le b-2} W(b-2-z, z), \delta v^A(b-1) \right\},$$

and, finally, for $b_1 = b_2 = 0$ it is

$$TW(0,0) = 2p^2 + \delta p^2 \max_{0 \le z \le 2} W(2-z, z) \tag{17}$$
$$+2p(1-p) + 2p(1-p) \max \left\{ 1 + \delta W(0,0), \delta v^A(1) \right\}.$$

In this case the Bellman equation can be solved and the social-welfare-maximizing strategies computed.

**Proposition 2.** *For $p < 1/2$ and $\delta < 1$ large enough, full sharing is socially optimal in the game $\mathbf{G_s}$, and the social welfare function is the symmetric function $W^s_\delta$ which solves, for all $b_2 \le b_1$,*

$$W^s_\delta(b_1, b_2) = \begin{cases} W^o_\delta(b, b) & \text{where } b = (b_1 + b_2)/2, \quad \text{if } b_1, b_2 \ge 0 \\ v^A(b_1) & \text{if } b_2 = -1. \end{cases}$$

The socially optimal strategy profile is not unique because any strategy profile which implements simple sharing on the diagonal before any individual dies does as well. For instance, a strategy profile in which no sharing takes place until one or more individual's stock hits zero, and then implements full sharing, does as well as a strategy which implements full sharing at the outset.

**Corollary 1.** *If the strategy profile is socially optimal, then no individual dies in a period in which, at the end, the other individual has a positive stock.*

# 4 Equilibrium

Our games are stochastic games with perfect monitoring. In this section we show what payoff profiles can and cannot be achieved in subgame-perfect equilibria (SPEs).

Stationary strategies in this game are assignments of rules to states; for instance, simple sharing at every state, or wealth-based sharing at every nonboundary state and autarky on the boundary. We will prove two kinds of results: that a particular outcome can be achieved in a Nash equilibrium (using the rules described above), and that a particular outcome cannot be achieved in any Nash equilibrium. We will prove results of the first kind by embedding the desired outcome in a autarkic (grim) trigger; for instance, "Use simple sharing until the first defection, and thereafter behave autarkically." We will see that not only is autarkic play an equilibrium, but also that the autarkic payoffs are the greatest lower bound on the set of equilibrium payoffs. For the strategies we consider, the only deviations that can pay are those in which an individual adds a unit to his inventory when the strategy calls for sharing. One simply compares the value of deviating in the current state and continuing in autarky—the best response to autarky—with the value of continuing on according to the equilibrium strategy forever. Proving the second kind of result proceeds in the same fashion, since the autarkic trigger is the worst punishment that can be inflicted on an individual. So, if the value of a deviation against an autarkic trigger is positive, then it will be positive against any conceivable equilibrium punishment strategy. Consequently, no punishment hurts enough to stop the individual from deviating.

## 4.1 The Curse of Poverty

The analysis begins with the bad news. The first observation, though, is a rather obvious one that holds true in both the flow and the stock game:

**Proposition 3.** *For all $p \in (0, 1)$ and all discount factors $\delta \in (0, 1)$, autarky—no sharing—is an equilibrium in games $\mathbf{G_o}$ and $\mathbf{G_s}$. The autarkic payoff is also the maximin payoff at this equilibrium of both games for all parameters.*

In particular, autarky is an equilibrium in an unproductive society for which $p < 1/2$. Unfortunately, in this case this is also the only equilibrium. This is the content of the following theorem, which justifies the second part of this paper's epigraph.

**Theorem 1.** *For $p < 1/2$ and $\delta$ sufficiently large, autarky is the only SPE in both $\mathbf{G_o}$ and $\mathbf{G_s}$.*

The theorem identifies a technological poverty trap: Societies, where individuals are doomed to die soon, will never develop altruistic behavior, even if individuals are very patient ($\delta \to 1$). The term "technological" here refers to the success parameter $p < 1/2$ that has a devastating effect on equilibrium behavior when it is too small. Intuitively, the high chances of the partner dying soon destroy the incentives for gift exchange—even if inventory holdings are high. If it is never likely that the favor will not be returned, there is no incentive to help out, whatever the inventory is. Patience ($\delta \to 1$) cannot compensate for this, because as long as $p < 1/2$ the chances of long run survival are zero. In this sense societies with too low productivity are doomed.

This result stands in striking contrast to the findings in favor of reciprocal altruism in repeated games. In our games, unlike in repeated games, there is no folk theorem. The presence of a state variable that affects the physical constraints of the system as it evolves destroys the logic of stationarity that the folk theorems exploit. The welfare contrast between our inventory game and repeated games, and the consequent failure of reciprocal altruism throughout the state space, is a central result of our paper.

The combination of Proposition 3 and Theorem 1 provides a full description of equilibrium behavior in poor societies, $p < 1/2$, if a disappointing one: There is a unique equilibrium that involves autarkic behavior at all inventory states. As there is nothing more to be said about the case with $p < 1/2$, we now turn to the case of higher productivity, $p > 1/2$.

### 4.2 The Blessings of Wealth

The case with $p > 1/2$ is a different world, because now the prospects of long-run survival provide incentives for cooperation. Figure 2 depicts the probability $\Pr\{\tau_i(\omega; \sigma, (0,0)) \to \infty\}$ as a function of the technology parameter $p \in (0,1)$ for the social optimum, equilibrium with simple sharing, and autarky. It shows that only for $p > 1/2$ there is a chance of long-run survival—and a significantly better one with sharing.

The extreme negative result of Theorem 1 is due to the certain prospect of collapse. There is no long run in which the patient individual can realize the benefits of previous cooperation. When $p \geq 1/2$, however, the possibility of an infinite long run emerges, and so one would expect the calculation to change. And it does. We will see that equilibria socially better than altruism exist, but maximal welfare is not achievable.

**Theorem 2.** *In both* $\mathbf{G_o}$ *and* $\mathbf{G_s}$ *no welfare-maximizing equilibrium exists for* $\delta$ *sufficiently close to 1.*

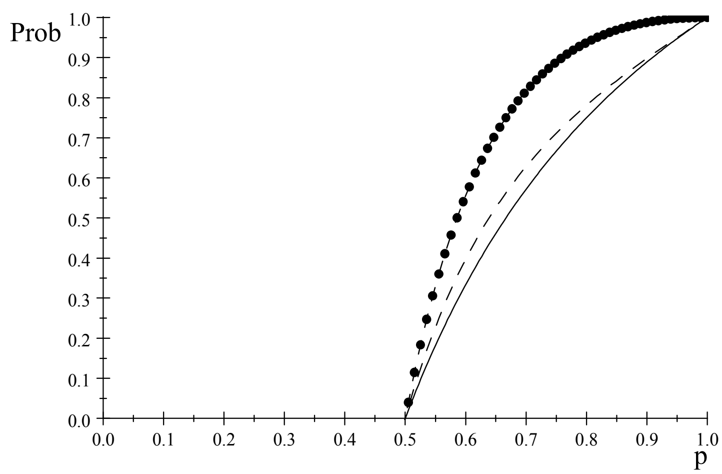



Figure 2: The probability of long-run survival as a function of $p \in (0, 1)$ for the social optimum (dash-dotted), equilibrium with simple sharing (dashed), and autarky (solid).

The theorem identifies a social poverty trap: Efficient behavior is never an equilibrium, independently of technological ($p$) or preference ($\delta$) parameters. Corollary 2 below shows that this property is a "poverty trap" because it affects the needy. Even at equilibria that involve a fair amount of sharing, the poorest will be excluded from the benefits of reciprocity. In particular, the following observation holds for all equilibria, both for the flow game $\mathbf{G_o}$ and the stock game $\mathbf{G_s}$: Sharing never occurs when it is needed most.

**Corollary 2.** *At any equilibrium of $\mathbf{G_o}$ or $\mathbf{G_s}$, when both inventories are depleted, $b_1 = b_2 = 0$, the equilibrium behavior is autarkic.*

Although we cannot find socially optimal equilbria, we can construct equilibria for the case $p > 1/2$ that do much better than autarky. This stands in contrast with $p < 1/2$. The higher productivity, larger $p$, allows for multiple welfare-ranked equilibrium.

We begin with the flow game $\mathbf{G_o}$ and with the rule of simple sharing: A successful individual gives one unit to the other if and only if the other failed.

**Proposition 4.** *For $p > 1/2$ and $\delta$ sufficiently large, simple sharing at all interior inventory states $(b_1, b_2) \gg 0$, and autarky whenever either $b_1 = 0$ or $b_2 = 0$ or both, is a SPE for the flow game $\mathbf{G_o}$.*

The proof of Proposition 4 computes the average discounted value by exploiting the fact that any interior $45^o$ line in $\mathcal{B}$ is invariant under simple sharing. Not surprisingly, therefore, in the interior the value $w_i$ has the form

$$w_i(b_1, b_2) = 1 - c\lambda^{\min\{b_1, b_2\}}, \tag{18}$$

where $c$ is a coefficient determined from boundary conditions that depends on the difference $b_1 - b_2$ and $\lambda$ is the root from (11). Hence, under simple sharing what counts is the minimum among the two invertory stocks. This is a symptom of the dependence on the sharing partner at such an equilibrium. Along the axes (where $b_1 = 0$ or $b_2 = 0$), though, equilibrium behavior is autarkic—for the reason identified in Theorem 2.

The autarkic behavior at the boundary is the reason why the sharing region matters. Even though all interior $45^o$ lines are invariant, when the boundary is hit, the process may move to a different $45^o$ line—and then it matters what equilibrium prescribes there. Therefore, we cannot directly conclude from Proposition 4 that there are many SPEs with simple sharing in different regions. Changing the sharing region will alter the value function. Yet, from a boundary point the (autarkic) process can only hit two inventory states at which both individuals have survived: If both succeed, they return to the inventory state, where they left the sharing region; if the rich individual failed and the poor succeeded, they hit a $45^o$ line which is closer to the principal diagonal than the one from which they left the sharing region. Therefore, the region between any $45^o$ line and the principal diagonal is invariant under simple sharing and the value depends only on what happens in this region. It follows that we can choose two $45^o$lines at different sides on the principal diagonal: Simple sharing in the interior and between these two $45^o$ lines will constitute a SPE.

Simple sharing in various regions and autarky are not the only equilibria for the flow game $\mathbf{G_o}$ when $p > 1/2$, though. Recall that *wealth-based sharing* refers to the rule where one individual gives one unit to the other if and only if the donor is successful and at least as rich as the recipient. Clearly, under this rule the donor is never better off than with simple sharing. For, in all contigencies where she gives under simple sharing, she also does under wealth-based sharing; but she also gives when both are successful and does not receive when the poor individual succeeded and the rich failed. On the other hand, an equilibrium with wealth-based sharing is capable of reaching the principal diagonal, where welfare is maximal for a fixed aggregate inventory stock.

Two qualifications come with an equilibrium that involves wealth-based sharing. First, as with simple sharing, at the boundary equilibrium entails autarkic behavior. Second, again like with simple sharing, there are many equilibria with wealth-based sharing. For, once again under wealth-based sharing the area between the principal diagonal and the $45^o$ line through the current inventory state is invariant. Therefore, the value of this strategy does not depend on what happens outside this region. As a consequence, any $45^o$ line may form the boundary of a sharing region.

**Proposition 5.** *For $p > 1/2$ and $\delta$ sufficiently large there is $M > 0$ such that there is a SPE with wealth-based sharing at all inventory states $(b_1, b_2) \in \mathcal{B}$ that satisfy $(b_1, b_2) \gg 0$ and*

$$\max\{b_1, b_2\} \leq \min\{b_1, b_2\} + 2M,$$

*and autarkic behavior outside this region.*

We suspect that the bound $M > 0$ is in fact infinity, but are so far unable to prove it. If this conjecture is true, then the maximal sharing region for wealth-based sharing is the same as for simple sharing: All of $\mathcal{B}$, except for the boundary where at least one individual runs out of stock.

The flow model treats inventory stocks like inalienable belly weight, illiquid machinery, or non-marketable housing. Things change when inventory stocks become transferable. If the needy can be given from the stock of the partner, private stocks become public. As long as agents are willing to give, all that counts is the aggregate inventory stock. Provided that $b_1 + b_2 \geq 2$ both individuals can be guaranteed survival, at least for another period. Of course, if $b_1 + b_2 = 0$, then cooperation cannot be sustained by Corollary 2.

Guaranteed survival everywhere except at the origin ($b_1 = b_2 = 0$) is indeed a SPE outcome in the stock game because of a new option that emerges when stocks are transferable. For, in the stock game inventories can be transferred *ex-ante*, prior to production. If transfers would only be allowed ex-post, after production has realized, it would not be incentive compatible to share at the boundary where either $b_1 = 0$ or $b_2 = 0$. (This follows from Theorem 2.) The possibility of ex-ante sharing improves welfare substantially.

**Proposition 6.** *For $p > 1/2$ and $\delta$ sufficiently large any strategy rule with ex-ante transfers that guarantees survival of both individuals everywhere except at the origin $(b_1, b_2) = (0, 0)$ constitutes a SPE for the stock game $\mathbf{G_s}$.*

Sharing cannot be incentive-compatible at the origin because at this inventory state only ex-post transfers are possible. And those will not occur in equilibrium by Corollary 2.

There are many equilibrium rules that support the outcomes of Proposition 6. For instance, at any inventory state other than the origin, if one individual is richer than the other, the rich transfers half the excess of her inventory over the poor's to the poor. That is, the two "jump" from any (non-zero) inventory state to the principal diagonal. Alternatively, at any inventory state and before production realizes, if one individual's inventory is depleted and the other's not, the rich transfers precisely one unit to the poor—a "charity" rule.

There are also rules that involve both ex-ante and ex-post transfers and support an equilibrium with the same value as in Proposition 6 (see (42) in the appendix). For instance, if one is richer than the other, the rich transfers half the excess of her inventory over the other's to the poor ex-ante and ex-post the simple sharing rule is applied (except at the origin). Since that keeps inventories at the principal diagonal, it must be equilibrium behavior. A less dramatic rule that also supports equilibrium is as follows. Ex-ante the rich individual (if there is one) transfers one unit to the poor and ex-post simple sharing applies, except at the origin. A "credit arrangement" of the following sort also supports equilibrium: Ex-ante the rich (if there is one) transfers one unit to the poor, and ex-post the poor returns the unit if and only if she was successful.

## 4.3 Welfare Losses

At many equilibria with sharing, the regions where no sharing occurs appear small compared to the whole integer lattice $\mathcal{B}$. On the other hand, no sharing obtains typically at the boundary, where the risk of death is largest. This raises a question about the quatitative implications of equilibrium as compared to the social optimum: How much welfare is lost in equilibrium?

To compute the minimum welfare losses at equilibrium we proceed as follows. For the case $p > 1/2$ any equilibrium with sharing yields at best

$$w(b, b) = 1 - \frac{(1-p)\left(1 + p\mu^2\right)}{1 - \delta p^2 \lambda} \lambda^b,$$

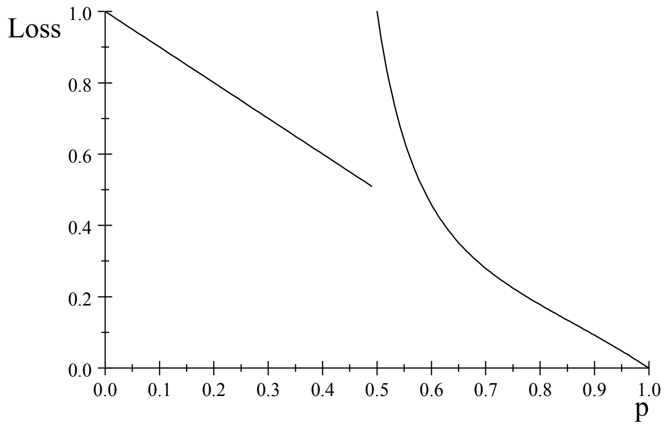



Figure 3: The relative welfare loss $\Delta(0)$ as a function of $p \in (0,1)$.

where $\mu$ and $\lambda$ are as in (4) and (11), along the principal diagonal. Comparing this to the welfare function $W_\delta^o(b,b)$ from (10) for the limiting case $\delta \to 1$ yields the measure

$$\Delta_o(b) = \lim_{\delta \nearrow 1} \left(1 - \frac{2w(b,b)}{(1-\delta) W_\delta^o(b,b)}\right) \tag{19}$$

of welfare losses for the flow game $\mathbf{G_o}$. Similarly, for the stock game $\mathbf{G_s}$ the limit

$$\Delta_s(b) = \lim_{\delta \nearrow 1} \left(1 - \frac{2\bar{v}^S(2b)}{(1-\delta) W_\delta^s(b,b)}\right) \tag{20}$$

(where $\bar{v}^S(b_1+b_2)$ is given in (42) in the appendix) provides a measure of the relative welfare loss in equilibrium as compared to the social optimum.

Yet, (18) together with (28) and (42) in the appendix imply that on the principal diagonal $\bar{v}^S(2b) = w(b,b)$, and Proposition 2 implies that $W_\delta^o(b,b) = W_\delta^s(b,b)$ for all $b = 0,1,...$ Therefore, $\Delta_o(b) = \Delta_s(b)$ for all $b = 0,1,...$ and the common measure of relative welfare losses may simply be denoted by $\Delta(b)$ for both the flow and the stock game. Using (5) and (12), for the case $p > 1/2$ the relative welfare loss may be expressed as

$$\Delta(b) = \frac{(1-p)^{2b+1}\left(p^2 + (1-p)^3\right)}{\left(1-(1-p)^2\right)\left(p^{2b+1} - (1-p)^{2b+2}\right)}. \tag{21}$$

In the case $p < 1/2$ the best equilibrium is autarky, both for $\mathbf{G_o}$ and for $\mathbf{G_s}$ by Theorem 1. Consequently, for $p < 1/2$,

$$\Delta(b) = \lim_{\delta \nearrow 1} \left(1 - \frac{v^A(b)}{W_\delta^o(b,b)}\right) = \frac{p^{b+1}(1-p)^{b+1} - (1-p)^{2b+2}}{p^{2b+2} - (1-p)^{2b+2}}, \tag{22}$$

again using (5) and (12).

Figure 3 depicts the relative welfare loss $\Delta(0)$ as a function of $p \in (0,1)$ evaluated at the origin $b_1 = b_2 = 0$. Not surprisingly, in both regimes ($p < 1/2$ and $p > 1/2$) the relative welfare loss is decreasing in $p$. But it is dramatic, since close to $p = 0$ and $p = 1/2$ almost all the potential welfare is lost. Around $p = 3/4$ about 22 percent of the potential welfare is lost in equilibrium; close to $p = 1/4$ it is about 75 percent. When $b$ grows, this becomes even worse for $p < 1/2$, but less pronounced for $p > 1/2$. Qualitatively the picture stays the same, though.

# 5 Conclusions

In this paper we study the incentives for cooperation among otherwise selfish individuals struggling for survival in a non-stationary environment—technically speaking a stochastic game. The results concur with widely held views about the development of societies. If productivity is very low and thus individuals see no chance of long-run survival, the only equilibrium is autarky, irrespective of whether or not belongings can be transferred (i.e., both in $\mathbf{G_o}$ and in $\mathbf{G_s}$). In this case ($p < 1/2$) any effort for reform is doomed, and welfare is minimal.

If productivity improves ($p > 1/2$), the picture changes. Even though autarky remains as one equilibrium, if policy can engineer a switch of equilibrium there are substantial welfare gains to be had. There are multiple equilibria, some of which entail significantly higher welfare as compared to autarky. The path to the blessings of wealth is not unique, though. Both when stocks can and cannot be transferred, there are several arrangements that generate welfare gains and are supported as equilibria. Yet, even the best among those do not achieve the social optimum. When bad luck hits, cooperation ceases to be incentive compatible. Even the best equilibria entail no sharing when it is needed most.

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# A Appendix: Proofs

This appendix is meant **for online publication**. For the reader's convenience we repeat the statements proved in the appendix. Yet, to begin with an auxiliary result is introduced.

**Lemma 1.** *For all* $(\delta, p) \in (0,1)^2$ *it holds that* $0 < \mu^2 < \lambda < \mu < \delta$*, where* $\mu$ *and* $\lambda$ *are defined in (4) and (11).*

*Proof.* That $\mu > 0$ is obvious from (4), and that $\mu < \delta$ follows from

$$1 - \sqrt{1 - 4\delta^2 p\,(1-p)} < 2\delta^2 p \Leftrightarrow 1 - 4\delta^2 p + 4\delta^4 p^2 < 1 - 4\delta^2 p\,(1-p) \Leftrightarrow \delta < 1.$$

That $0 < \mu^2 < \lambda$ follows from $\mu > 0$, (4), (11), and from

$$\begin{aligned}
&\lambda = \frac{1 - 2\delta p\,(1-p) - \sqrt{1 - 4\delta p\,(1-p)}}{2\delta p^2} > \mu^2 = \frac{\left(1 - \sqrt{1 - 4\delta^2 p\,(1-p)}\right)^2}{4\delta^2 p^4} \Leftrightarrow \\
&1 - \sqrt{1 - 4\delta p\,(1-p)} > \frac{1}{\delta} - \frac{1}{\delta}\sqrt{1 - 4\delta^2 p\,(1-p)} \Leftrightarrow \\
&\quad \sqrt{1 - 4\delta^2 p\,(1-p)} > 1 - \delta + \delta\sqrt{1 - 4\delta p\,(1-p)} \Leftrightarrow \\
&1 - 2\delta p\,(1-p) > \sqrt{1 - 4\delta p\,(1-p)} \Leftrightarrow \lambda > 0.
\end{aligned}$$

It remains to show that $\lambda < \mu$. To that end observe that

$$\frac{\partial \mu}{\partial \delta} = \frac{\mu}{\delta\sqrt{1 - 4\delta^2 p\,(1-p)}} \longrightarrow_{\delta \nearrow 1} \begin{cases} \frac{1}{1-2p} & \text{if} \quad p < \frac{1}{2} \\ \\ \frac{1-p}{p(2p-1)} & \text{if} \quad p > \frac{1}{2} \end{cases} \tag{23}$$

and

$$\frac{\partial \lambda}{\partial \delta} = \frac{\lambda}{\delta\sqrt{1 - 4\delta p\,(1-p)}} \longrightarrow_{\delta \nearrow 1} \begin{cases} \frac{1}{1-2p} & \text{if} \quad p < \frac{1}{2} \\ \\ \frac{(1-p)^2}{p^2(2p-1)} & \text{if} \quad p > \frac{1}{2} \end{cases} \tag{24}$$

which implies that whenever (for fixed $p$) there is some $\delta_0 \in (0,1)$ such that $\lambda \geq \mu$, then continuity and that $\partial\lambda/\partial\delta > \partial\mu/\partial\delta$ at $\delta = \delta_0$ imply that $\lambda > \mu$ for all $\delta > \delta_0$. But the latter contradicts $\lim_{\delta \nearrow 1} \lambda \leq \lim_{\delta \nearrow 1} \mu$ from (5) and (12). Hence, $\lambda < \mu$ for all $\delta \in (0,1)$. □

**Proposition 1.** *The policy of wealth-based sharing off the diagonal of* $\mathcal{B}$*, and simple sharing on the diagonal, is socially optimal in the flow game* $\mathbf{G_o}$.

*Proof.* Let $\mathcal{W}$ denote the set of all functions $W : \mathcal{B} \to \mathbb{R}_+$ that satisfy, for all $(b_1, b_2) \in \mathcal{B}$,

**(a)** $W$ is symmetric and monotone increasing, i.e., $W(b_1, b_2) = W(b_2, b_1)$ and if $(b_1', b_2') \geq (b_1, b_2)$ with $(b_1', b_2') \neq (b_1, b_2)$ then $W(b_1', b_2') > W(b_1, b_2)$, for all $(b_1', b_2') \in \mathcal{B}$;

**(b)** if $0 \leq b_1 < b_2$ then $W(b_1 + 1, b_2 - 1) \geq W(b_1, b_2)$;

**(c)** $W(0, b_2) \geq v^A(b_2 + 1) - 1/\delta$.

Define the Bellman operator $T : \mathcal{W} \to \mathcal{W}$ by (7), (8), and (9). The maximum welfare for the flow game $\mathbf{G_o}$ is the unique fixed point $W_\delta^o$ of $T$ on the function space $\mathcal{W}$. To show existence and uniqueness, we first claim that if $W \in \mathcal{W}$, then $TW \in \mathcal{W}$, that is, $\mathcal{W}$ is closed under the operator $T$.

If $W$ is symmetric and monotone increasing, then it is straightforward to show by direct substitution that $TW$ is symmetric and monotone increasing, which verifies (a). As for (b) and (c), observe first that if $0 < b_1 < b_2$, setting $0 \leq b_1' = b_1 - 1 < b_2' = b_2 + 1$ implies by (b) for $W \in \mathcal{W}$ that

$$W(b_1' + 1, b_2' - 1) = W(b_1, b_2) \geq W(b_1', b_2') = W(b_1 - 1, b_2 + 1). \qquad (25)$$

Therefore, if $W \in \mathcal{W}$ and $0 < b_1 < b_2$, then

$$\begin{aligned} TW(b_1, b_2) = 2 + \delta(1-p)^2 W(b_1 - 1, b_2 - 1) + \delta p^2 W(b_1 + 2, b_2) \\ + \delta p(1-p) W(b_1 + 1, b_2 - 1) + \delta p(1-p) W(b_1, b_2) \end{aligned}$$

by (b) and (25). This implies that

$$\begin{aligned} TW(b_1 + 1, b_2 - 1) - TW(b_1, b_2) = {} & \delta(1-p)^2 W(b_1, b_2 - 2) \\ & + \delta p^2 \max\{W(b_1 + 2, b_2), W(b_1 + 3, b_2 - 1)\} \\ & + \delta p(1-p) \max\{W(b_1 + 2, b_2 - 2), W(b_1 + 1, b_2 - 1)\} \\ & - \delta(1-p)^2 W(b_1 - 1, b_2 - 1) \\ & - \delta p^2 W(b_1 + 2, b_2) - \delta p(1-p) W(b_1, b_2) \geq 0, \end{aligned}$$

because all differences for the cases, which occur with probabilities $(1-p)^2$, $p^2$, and $p(1-p)$ respectively, are nonnegative by (b), $W \in \mathcal{W}$, and the definition of a maximum. If $b_1 = 0$ but $b_2 > 1$, then

$$
\begin{aligned}
TW\left(1, b_2 - 1\right) - TW\left(0, b_2\right) = \delta\left(1-p\right)^2 &\left[W\left(0, b_2 - 2\right) - v^A\left(b-1\right) + \frac{1}{\delta}\right] \\
&+ \delta p^2\left[\max\left\{W\left(3, b_2 - 1\right), W\left(2, b_2\right)\right\} - W\left(2, b_2\right)\right] \\
&+ \delta p\left(1-p\right)\left[W\left(2, b_2 - 1\right) - W\left(0, b_2\right)\right] \geq 0,
\end{aligned}
$$

because the first term in square brackets is nonnegative by (c) and $W \in \mathcal{W}$, the second term in square brackets is nonnegative by (b) and $W \in \mathcal{W}$ for $b_2 > 2$ and by $W\left(2,2\right) \geq W\left(3,1\right) = W\left(1,3\right)$ from (a) and (b) and $W \in \mathcal{W}$ for $b_2 = 2$, and the third term is nonnegative by (b) and $W \in \mathcal{W}$. Finally, if $0 = b_1 < b_2 = 1$, then by $W \in \mathcal{W}$ and symmetry $TW\left(1,0\right) = TW\left(0,1\right) = 0$. Therefore, if $W \in \mathcal{W}$, then $TW$ satisfies (a) and (b).

It remains to show that if $W \in \mathcal{W}$, then $TW$ satisfies (c). Because for $b_2 > 0$ and $W \in \mathcal{W}$ by (b), (c) for $W$, and (25)

$$
\begin{aligned}
TW\left(0, b_2\right) = {} & 2 + \delta\left(1-p\right)^2\left[v^A\left(b_2 - 1\right) - \frac{1}{\delta}\right] + \delta p^2 W\left(2, b_2\right) \\
& + \delta p\left(1-p\right) W\left(1, b_2 - 1\right) + \delta p\left(1-p\right) W\left(0, b_2\right) \\
& \geq 2 + \delta\left(1-p\right)^2\left[v^A\left(b_2 - 1\right) - \frac{1}{\delta}\right] + \delta p^2 W\left(2, b_2\right) \\
& + 2\delta p\left(1-p\right) W\left(0, b_2\right) \geq 2 + \delta\left(1-p\right)^2\left[v^A\left(b_2 - 1\right) - \frac{1}{\delta}\right] \\
& + \delta p^2 W\left(0, b_2 + 2\right) + 2\delta p\left(1-p\right) W\left(0, b_2\right) \geq 2 + \\
& + \delta\left(1-p\right)^2\left[v^A\left(b_2 - 1\right) - \frac{1}{\delta}\right] + \delta p^2\left[v^A\left(b_2 + 3\right) - \frac{1}{\delta}\right] \\
& + 2\delta p\left(1-p\right)\left[v^A\left(b_2 + 1\right) - \frac{1}{\delta}\right] = 1 + \delta\left(1-p\right)^2 v^A\left(b_2 - 1\right) \\
& + \delta p^2 v^A\left(b_2 + 3\right) + 2\delta p\left(1-p\right) v^A\left(b_2 + 1\right),
\end{aligned}
$$

$TW$ will satisfy (c) if

$$
\begin{aligned}
1 + \delta\left(1-p\right)^2 v^A\left(b_2 - 1\right) + \delta p^2 v^A\left(b_2 + 3\right) + 2\delta p\left(1-p\right) v^A\left(b_2 + 1\right) \\
\geq v^A\left(b_2 + 1\right) - \frac{1}{\delta}.
\end{aligned}
$$

Using (3) the latter is equivalent to

$$1-\delta \geq \mu^{b_2}\left[\delta p^2\mu^4-(1-2\delta p(1-p))\mu^2+\delta(1-p)^2\right],$$

where $\mu$ is given in (4). Since $\mu \in (0,1)$ by Lemma 1, this inequality will be true for all $b_2 > 0$ if it is true for $b_2 = 1$, that is, if

$$1-\delta-\mu\left[\delta p^2\mu^4-(1-2\delta p(1-p))\mu^2+\delta(1-p)^2\right] \geq 0.$$

Using that $\mu$ solves $\mu = \delta p\mu^2 - \delta(1-p)$ the left hand side of the last inequality can be rewritten as

$$\begin{aligned} &1-\delta-\mu\left[\delta p^2\mu^4-(1-2\delta p(1-p))\mu^2+\delta(1-p)^2\right] \\ =\, &1-\delta-\mu\left[\delta p^2\left(\frac{\mu}{\delta p}-\frac{1-p}{p}\right)^2-(1-2\delta p(1-p))\left(\frac{\mu}{\delta p}-\frac{1-p}{p}\right)+\delta(1-p)^2\right] \\ =\, &(1-\delta)\left(1-\frac{\mu^3}{\delta}\right)=(1-\delta)^2 v^A(2) \geq 0 \end{aligned}$$

because $\mu < \delta$ by Lemma 1. This takes care of the case $b_2 > b_1 = 0$. For $b_1 = b_2 = 0$ by $W \in \mathcal{W}$, (a), (b), and (c) for $W$,

$$\begin{aligned} TW(0,0) &= 2\left(1-(1-p)^2\right)+\delta p^2 W(1,1)+2\delta p(1-p)W(0,0) \\ &\geq 2\left(1-(1-p)^2\right)+\delta p^2 W(0,2)+2\delta p(1-p)W(0,0) \\ &\geq 1-(1-p)^2+\delta p^2 v^A(3)+2\delta p(1-p)v^A(1) \\ &= \frac{1}{1-\delta}\left(1-\frac{\mu^2}{\delta^2}\right) > v^A(1)-\frac{1}{\delta}, \end{aligned}$$

where the last inequality follows from $\mu^2 < \delta$ by Lemma 1. Therefore, $TW(0,0) \geq v^A(1) - 1/\delta$ and $TW$ satisfies (c) as well.

Together the above arguments imply that $T$ is a selfmap $T : \mathcal{W} \to \mathcal{W}$. The uniform metric defined by

$$\left\|W,\hat{W}\right\| = \sup_{(b_1,b_2)\in\mathcal{B}}\left|W(b_1,b_2)-\hat{W}(b_1,b_2)\right|,$$

for all $W, \hat{W} \in \mathcal{W}$, induces a topology on $\mathcal{W}$ that makes it a complete metric space, as all $W \in \mathcal{W}$ are bounded. Since $TW$, besides the constant, is a linear combination of values of $W$ with positive coefficients that sum to at most $\delta \in$

$(0,1)$, it follows that $\left\|TW, T\hat{W}\right\| \leq \delta \left\|W, \hat{W}\right\|$ for all $W, \hat{W} \in \mathcal{W}$. Therefore, $T$ is a contraction mapping. An application of Banach's fixed point theorem (Aliprantis and Border, 1999, Theorem 3.48, p. 95) now yields the existence of the unique welfare function $W_\delta^o$. The statement of the Proposition follows from (b) and (25). □

**Proposition 2.** *For $p < 1/2$ and $\delta < 1$ large enough, full sharing is socially optimal in the game $\mathbf{G_s}$, and the social welfare function is the symmetric function $W_\delta^s$ which solves, for all $b_2 \leq b_1$,*

$$W_\delta^s(b_1, b_2) = \begin{cases} W_\delta^o(b, b) & \text{where } b = (b_1 + b_2)/2, \quad \text{if } b_1, b_2 \geq 0 \\ v^A(b_1) & \text{if } b_2 = -1. \end{cases}$$

*Proof.* We need to be shown that $W_\delta^s(b_1, b_2) = W_\delta^o((b_1 + b_2)/2, (b_1 + b_2)/2)$ is the unique fixed point of the Bellman operator given by (15), (16), and (17). The first step concerns the terms multiplied by $(1-p)^2$ and $2p(1-p)$ in (16), in particular $2 + \delta W_\delta^o(b_2/2, b_2/2) > 1 - \delta v^A(b_2 + 1)$ for all $b_2 = 0, 2, ..., p > 1/2$, and $\delta$ sufficiently close to 1. Using (10) and (3) this inequality is equivalent to

$$1 + \frac{2\delta}{1-\delta}\left(1 - \frac{\lambda^{b_2/2+1}}{\delta}\right) > \frac{\delta}{1-\delta}\left(1 - \frac{\mu^{b_2+2}}{\delta}\right)$$
$$\Leftrightarrow 1 > 2\lambda^{b_2/2+1} - \mu^{b_2+2}.$$

The last inequality follows from

$$\lim_{\delta \nearrow 1}\left(2\lambda^{b_2/2+1} - \mu^{b_2+2}\right) = \frac{(1-p)^{b_2+2}}{p^{b_2+2}} < 1 \Leftrightarrow \frac{1}{2} < p$$

by (12) and (5). It now follows that

$$TW(0, b_2) = 2 + \delta p^2 \max_{0 \leq z \leq b_2+2} W(b_2 + 2 - z, z)$$
$$+ 2\delta p(1-p) \max_{0 \leq z \leq b_2} W(b_2 - z, z) + \delta(1-p)^2 \max_{0 \leq z \leq b_2-2} W(b_2 - 2 - z, z)$$

and

$$TW(0,0) = 2\left(1 - (1-p)^2\right) + \delta p^2 \max_{0 \leq z \leq 2} W(2 - z, z) + 2\delta p(1-p) W(0,0).$$

To see that $W_\delta^s(b_1, b_2) = W_\delta^o((b_1 + b_2)/2, (b_1 + b_2)/2)$ solves the Bellman equations, notice first that that $W_\delta^o((b_1 + b_2)/2, (b_1, +b)/2) \geq W_\delta^o(b_1 + b_2 - z, z)$

for all $z = 0, ..., b_1 + b_2$ by property (b) and (25) of the function $W_\delta^o \in \mathcal{W}$. Therefore, for $(b_1, b_2) \gg 0$ with $B = b_1 + b_2$,

$$
\begin{aligned}
TW_\delta^o (B/2, B/2) = 2&+\delta (1-p)^2 W_\delta^o (B/2-1, B/2-1) \\
+2\delta p (1-p) W_\delta^o (B/2, B/2) &+ \delta p^2 W_\delta^o (B/2+1, B/2+1) \\
= \frac{2}{1-\delta} \left(1 - \frac{\lambda^{B/2+1}}{\delta}\right) &= W_\delta^o (B/2, B/2),
\end{aligned}
$$

using that $\lambda = \delta p^2 \lambda^2 + 2\delta p (1-p) \lambda + \delta (1-p)^2$. Likewise, for $(b_1, b_2) = (0, b_2)$ and $b_2 = 2, 4, ...$ we obtain

$$
\begin{aligned}
TW_\delta^o (b_2/2, b_2/2) &= 2 + \delta p^2 W_\delta^o (b_2/2+1, b_2/2+1) \\
+2\delta p (1-p) W_\delta^o (b_2/2, b_2/2) &+ \delta (1-p)^2 W_\delta^o (b_2/2-1, b_2/2-1) \\
&= W_\delta^o (b_2/2, b_2/2)
\end{aligned}
$$

by the same computation as before. Finally, at the origin

$$
\begin{aligned}
TW_\delta^o (0,0) &= 2 \left(1 - (1-p)^2\right) + \delta p^2 W_\delta^o (1,1) + 2\delta p (1-p) W_\delta^o (0,0) \\
&= \frac{2}{1-\delta} \left(1 - \frac{\lambda}{\delta}\right) = W_\delta^o (0,0),
\end{aligned}
$$

using (10), as required. Therefore, $W_\delta^o ((b_1+b_2)/2, (b_1+b_2)/2)$ is a fixed point of the Bellman operator. With the uniform metric it is easy to show, like in the previous proof, that $T$ is a contraction. Hence, the fixed point is unique by Banach's fixed point theorem. □

**Theorem 2.** *In both* $\mathbf{G_o}$ *and* $\mathbf{G_s}$ *no welfare-maximizing equilibrium exists for* $\delta$ *sufficiently close to 1.*

*Proof.* In both games, the social optimum require sharing along the diagonal beginning at $(0,0)$ whenever one person succeeds and the other does not. Thus the principle diagonal is invariant, and the welfare functions for both games are identical on the principal diagonal. We will show that sharing at the origin is not equilibrium behavior.

Suppose that $\sigma$ is an equilibrium strategy requiring sharing at the origin, and denote by $\nu_{i\sigma}(b)$ the expected future payoff to individual $i$ from equilibrium behavior at the symmetric inventory state $(b,b)$. A necessary condition for equilibrium is that for all $b$ and for both individuals $i = 1, 2$, $\nu_{i\sigma}(b) \geq v^A(b+1)$.

Adding over individuals and using the definition of the social optimum yields

$$W_\delta^h(b,b) \geq \nu_{1\sigma}(b) + \nu_{2\sigma}(b) \geq 2v^A(b+1),$$

where $h = o, s$ depending on the game (recall Proposition 2). The first equation is a welfare bound on any strategy, and the second is an incentive constraint, that deviation does not pay. In particular these inequalities must hold at the origin:

$$W_\delta^h(0,0) \geq \nu_{1\sigma}(0) + \nu_{2\sigma}(0) \geq 2v^A(1).$$

Yet, substituting (10) and (3) the first and the last inequality combine to

$$W_\delta^h(0,0) = \frac{2}{1-\delta}\left(1 - \frac{\lambda}{\delta}\right) \geq 2v^A(1) = \frac{2}{1-\delta}\left(1 - \frac{\mu^2}{\delta}\right),$$

which is equivalent to $\mu^2 \geq \lambda$, contradicting Lemma 1. It follows that $2v^A(1) > W_\delta^h(0,0) \geq \nu_{1\sigma}(0) + \nu_{2\sigma}(0)$. Hence, at least one individual has an incentive to deviate from sharing. $\square$

**Proposition 3.** *For all $p \in (0,1)$ and all discount factors $\delta \in (0,1)$, autarky—no sharing—is an equilibrium in games $\mathbf{G_o}$ and $\mathbf{G_s}$. The autarkic payoff is also the maximin payoff at this equilibrium of both games for all parameters.*

*Proof.* An individual, say individual 1, can guarantee himself the autarkic payoff regardless of what anyone else does. If the other individual is behaving autarkically, this is the best that individual 1 can do. Thus it is both maximin and Nash starting from any initial stock vector. Since it is Nash from any initial state, it is subgame perfect. $\square$

**Theorem 1.** *For $p < 1/2$ and $\delta$ sufficiently large, autarky is the only SPE in both $\mathbf{G_o}$ and $\mathbf{G_s}$.*

*Proof.* Consider first the flow game $\mathbf{G_o}$. We proceed indirectly, and suppose that (ex-post) sharing occurs at inventory state $(b_1, b_2) \in \mathcal{B}$ and that this is equilibrium behavior. Denote by $\nu_{i\sigma}(b_1, b_2)$ the expected future payoff to individual $i = 1, 2$ in this equilibrium $\sigma$. The equilibrium hypothesis implies that $\nu_{i\sigma}(b_1, b_2) \geq v^A(b_i + 1)$ for $i = 1, 2$, hence,

$$\nu_{1\sigma}(b_1, b_2) + \nu_{2\sigma}(b_1, b_2) \geq v^A(b_1 + 1) + v^A(b_2 + 1).$$

From the definition of $W_\delta^o$ it follows that $W_\delta^o(b_1,b_2) \geq \nu_{1\sigma}(b_1,b_2) + \nu_{2\sigma}(b_1,b_2)$; and from (25) it follows that

$$W_\delta^o\left(\frac{b_1+b_2}{2},\frac{b_1+b_2}{2}\right) \geq W_\delta^o(b_1,b_2) \geq \nu_{1\sigma}(b_1,b_2)+\nu_{2\sigma}(b_1,b_2) \\ \geq v^A(b_1+1)+v^A(b_2+1).$$

Using (6) and (13) for $p<1/2$, as $\delta$ goes to 1 the first and last inequality combine to

$$\frac{b_1+b_2+4p}{1-2p} \geq \frac{b_1+b_2+2+4p}{1-2p},$$

equivalently, $0 \geq 2$, a contradiction.

For the stock game $\mathbf{G_s}$, suppose to the contrary that for $p<1/2$ there is an equilibrium that improves upon autarky. Let $\nu_{i\sigma}(b_1,b_2)$ denote the value of this equilibrium $\sigma$ for individual $i$ at inventory state $(b_1,b_2)$. Since sharing at the origin $(b_1,b_2)=(0,0)$ would have to be ex-post, it is impossible by Corollary 2. Therefore, the strategy combination that ex-ante always equalizes the two stocks must do at least as well as $\sigma$. This strategy combination yields to each individual the expected payoff $v^S(b_1+b_2)$ that solves

$$v^S(B) = 1+\delta p^2 v^S(B+2)+2\delta p(1-p)v^S(B)+\delta(1-p)^2 v^S(B-2)$$

for each $B \equiv b_1+b_2 = 2,4,...$, with initial condition

$$v^S(0) = p+\delta p^2 v^S(2)+\delta p(1-p)v^A(1).$$

The bounded solution to this linear second-order difference equation is given by

$$v^S(B) = \frac{1}{1-\delta}\left(1-\left(\frac{(1-p)(p+\mu/\delta)}{1-\delta p^2\lambda}\right)\lambda^{B/2}\right), \tag{26}$$

where $\mu$ and $\lambda$ are as in (4) and (11), respectively. It follows that $\nu_{i\sigma}(b_1,b_2) \leq v^S((b_1+b_2)/2)$ for all $i$.

For $\sigma$ to be an equilibrium it is necessary that $\nu_{i\sigma}(b_1,b_2) \geq v^A(b_i+1)$ for all $i$ at all inventory states, where $\sigma$ prescribes sharing. This implies that $v^S(b_1+b_2) \geq v^A(b_i+1)$ for all $i=1,2$ at all states, where $\sigma$ prescribes sharing. Let $(b_1,b_2)\neq(0,0)$ be such a state, which exists by hypothesis. It follows from

$2v^S(b_1+b_2) \geq \nu_{i\sigma}(b_1+1)+\nu_{i\sigma}(b_2+1)$, (3), and (26) that

$$\mu^{b_1+2}+\mu^{b_2+2} \geq \frac{2(1-p)(\delta p+\mu)}{1-\delta p^2\lambda}\lambda^{(b_1+b_2)/2}.$$

At $\delta = 1$ the last inequality is an equality by (5) and (12). Taking derivatives with respect to $\delta$ for $p < 1/2$ yields

$$\begin{aligned}\frac{\partial\left(\mu^{b_1+2}+\mu^{b_2+2}\right)}{\partial\delta} &= (b_1+2)\,\mu^{b_1+1}\frac{\partial\mu}{\partial\delta}+(b_2+1)\,\mu^{b_2+1}\frac{\partial\mu}{\partial\delta} \\ &= \frac{(b_1+2)\,\mu^{b_1+2}+(b_2+2)\,\mu^{b_2+2}}{\partial\sqrt{1-4\delta^2 p(1-p)}} \rightarrow_{\delta\nearrow 1} \frac{b_1+b_2+4}{1-2p},\end{aligned}$$

using (23) and (5), for the left hand side, and

$$\begin{aligned}\frac{\partial\left(\frac{2(1-p)(\delta p+\mu)}{1-\delta p^2\lambda}\lambda^{(b_1+b_2)/2}\right)}{\partial\delta} &= \frac{(1-p)(\delta p+\mu)}{1-\delta p^2\lambda}(b_1+b_2)\,\lambda^{(b_1+b_2)/2-1}\frac{\partial\lambda}{\partial\delta} \\ +\frac{2(1-p)\left(p+\frac{\partial\mu}{\partial\delta}\right)\lambda^{(b_1+b_2)/2}}{1-\delta p^2\lambda} &+ \frac{2(1-p)(\delta p+\mu)\left(p^2\lambda+\delta p^2\frac{\partial\lambda}{\partial\delta}\right)\lambda^{(b_1+b_2)/2}}{(1-\delta p^2\lambda)^2} \\ = \frac{2p^2(1-p)(\delta p+\mu)\,\lambda^{(b_1+b_2)/2+1}}{(1-\delta p^2\lambda)^2} &+ \frac{2(1-p)\,\mu\lambda^{(b_1+b_2)/2}}{\delta(1-\delta p^2\lambda)\sqrt{1-4\delta^2 p(1-p)}} \\ +\frac{2p(1-p)\,\lambda^{(b_1+b_2)/2}}{1-\delta p^2\lambda} &+ \frac{2p^2(1-p)(\delta p+\mu)\,\lambda^{(b_1+b_2)/2+1}}{(1-\delta p^2\lambda)^2\sqrt{1-4\delta p(1-p)}} \\ &+\frac{(1-p)(\delta p+\mu)(b_1+b_2)\,\lambda^{(b_1+b_2)/2}}{\delta(1-\delta p^2\lambda)\sqrt{1-4\delta p(1-p)}},\end{aligned}$$

using (23) and (24), for the right hand side. By (5) and (12) the limit as $\delta$ goes to 1 of the last expression is

$$\frac{2p^2}{1-p^2}+\frac{2}{(1+p)(1-2p)}+\frac{2p}{1+p}+\frac{2p^2}{(1-p^2)(1-2p)}+\frac{b_1+b_2}{1-2p}=\frac{b_1+b_2+2}{1-2p}.$$

Since $0 < (b_1+b_2+2)/(1-2p) < (b_1+b_2+4)/(1-2p)$, the slope of the right hand side is strictly smaller than the slope of the left hand side by $p < 1/2$. Therefore, for $\delta < 1$ large enough $v^A(b_1+1)+v^A(b_2+1) > 2v^S(b_1+b_2)$, a contradiction.[2] □

**Proposition 4** *For $p > 1/2$ and $\delta$ sufficiently large, simple sharing at all inte-*

[2] Note that if a deviator $i$ can "steal" more than one unit, $\mu^{b_i+2}$ is *reduced* to $\mu^{b_i+2+k}$ for some $k > 0$ by Lemma 1, and the argument bites even more.

*rior inventory states $(b_1, b_2) \gg 0$, and autarky whenever either $b_1 = 0$ or $b_2 = 0$ or both, is a SPE for the flow game $\mathbf{G_o}$.*

*Proof.* Let $w_i(b_1, b_2)$ denote the normalized value (multiplied with $1-\delta$) of the strategy combination described in the statement. By symmetry $w_1(b_1, b_2) = w_2(b_2, b_1)$ and the subscript $i$ for the individual's identity can be dropped on the understanding that the first argument of the function $w : \mathcal{B} \to \mathbb{R}$ is $b_1$ and the second $b_2$, i.e., $w_1(b_1, b_2) = w(b_1, b_2)$ and $w_2(b_1, b_2) = w(b_2, b_1)$. Since we consider the even integer lattice, if $b_i = 0$ then $b_{3-i}$ is an even integer, for $i = 1, 2$.[3] The normalized value function $w$ satisfies, for $(b_1, b_2) \gg 0$,

$$\begin{aligned} w(b_1, b_2) = {} & 1 - \delta + \delta p^2 w(b_1 + 1, b_2 + 1) + \delta (1-p)^2 w(b_1 - 1, b_2 - 1) \\ & + 2\delta p (1-p) w(b_1 + b_2), \end{aligned}$$

for $(b_1, b_2) = (b_1, 0)$ with $b_1 > 0$,

$$\begin{aligned} w(b_1, 0) = {} & 1 - \delta + \delta p^2 w(b_1 + 1, 1) + \delta (1-p)^2 (1-\delta) v^A (b_1 - 1) \\ & + \delta p (1-p) w(b_1 - 1, 1) + \delta p (1-p)(1-\delta) v^A (b_1 + 1), \end{aligned}$$

for $(b_1, b_2) = (0, b_2)$ with $b_2 > 0$,

$$w(0, b_2) = (1-\delta) p + \delta p^2 w(1, b_2 + 1) + \delta p (1-p) w(1, b_2 - 1),$$

and for $(b_1, b_2) = (0, 0)$ it satisfies

$$w(0, 0) = (1-\delta) p + \delta p^2 w(1, 1) + \delta p (1-p)(1-\delta) v^A (1).$$

Let $\mu$ be as in (4), $\lambda$ as in (11), and define $\gamma > 0$ by

$$\gamma = \frac{\delta p (1-p) \lambda}{1 - \delta p^2 \lambda} = \frac{(1-p)\left[1 - 2\delta p (1-p) - \sqrt{1 - 4\delta p (1-p)}\right]}{p\left[1 + 2\delta p (1-p) + \sqrt{1 - 4\delta p (1-p)}\right]}.$$

The the bounded solution, for $(b_1, b_2) \gg 0$, is given by

$$w(b_1, b_2) = 1 - \left[1 - w(\max\{0, b_1 - b_2\}, \max\{0, b_2 - b_1\})\right] \lambda^{\min\{b_1, b_2\}} \quad (27)$$

[3] The result also holds on the odd integer lattice, though.

with boundary conditions, for $b = 2, 4, ...,$[4]

$$w(0,b) = 1 - \frac{1-p}{1-\delta p\lambda} - \left[1 - w(0,0) - \frac{1-p}{1-\delta p\lambda}\right]\gamma^{b/2},$$

and for $b = 2, 4, ...,$

$$w(b,0) = 1 - \frac{(1-p)\,\mu^{b+2}}{\delta(\mu - p\lambda)} - \left[1 - w(0,0) - \frac{(1-p)\,\mu^{3}}{\delta(\mu - p\lambda)}\right]\gamma^{b/2},$$

where the value at the origin is given by

$$w(0,0) = 1 - \frac{(1-p)\left(1+p\mu^2\right)}{1-\delta p^2\lambda}. \tag{28}$$

This can be verified by direct substitution, using that $\lambda = \delta p^2\lambda^2 + 2\delta p(1-p)\lambda + \delta(1-p)^2$, $\mu = \delta p\mu^2 + \delta(1-p)$, (3), and that $\left(1-\delta p^2\lambda\right)\left(\mu^2-\gamma\right) = \mu(\mu - p\lambda)$.

The incentive constraints for simple sharing in the interior of $\mathcal{B}$ at $b_1 > b_2 > 0$ are given by

$$w(b_1,b_2) \geq (1-\delta)\,v^A(b_1+1) \Leftrightarrow \mu^{b_1+2} \geq \delta\left[1 - w(b_1-b_2,0)\right]\lambda^{b_2} \tag{29}$$

and at $b_2 \geq b_1 > 0$ by

$$w(b_1,b_2) \geq (1-\delta)\,v^A(b_1+1) \Leftrightarrow \mu^{b_1+2} \geq \delta\left[1 - w(0,b_2-b_1)\right]\lambda^{b_1}. \tag{30}$$

To analyze the limiting case as $\delta$ goes to 1 we reparametrize by $q = (1-p)/p \in (0,1) \Leftrightarrow p = 1/(1+q) \in (1/2, 1)$ and use (5) and (12) for $p > 1/2$ to obtain

$$\lim_{\delta\nearrow 1}\gamma = \frac{(1-p)^3}{p^2(2-p)} = \frac{q^3}{1+2q},\ \lim_{\delta\nearrow 1} w(0,0) = \frac{2p-1}{p^2(2-p)} = \frac{\left(1-q^2\right)(1+q)}{1+2q},$$

and, for $b = 2, 4, ...,$ the limiting boundary values

$$\begin{aligned}\lim_{\delta\nearrow 1} w(0,b) &= \frac{2p-1}{p-(1-p)^2}\left[1 - \frac{(1-p)^{3b/2+3}}{p^{b+2}(2-p)^{b/2+1}}\right]\\ &= \frac{1-q^2}{1+q-q^2}\left[1 - \frac{q^{3b/2+3}}{(1+2q)^{b/2+1}}\right]\end{aligned}$$

[4] Note that whenever $(b_1, b_2) \in \mathcal{B}$ and $b_1 = 0$ but $b_2 > 0$, then $b_2 = 2, 4, ...$

and

$$\lim_{\delta\nearrow 1} w(b,0) = 1 - \left(\frac{1-p}{p}\right)^{b+2} - \left[\frac{(1-p)\left(1-p+p^2\right)}{p^2(2-p)} - \left(\frac{1-p}{p}\right)^3\right]\frac{(1-p)^{3b/2}}{p^b(2-p)^{b/2}}$$
$$= 1 - q^{b+2} - \left(1+q-2q^3\right)\frac{q^{3b/2+1}}{(1+2q)^{b/2+1}}$$

for all $b = 2, 4, ...$ With these at hand the incentive constraint (29) approaches, as $\delta$ goes to 1, for $b_1 > b_2 > 0$,

$$q^{b_1-b_2+2} \geq q^{b_1+2} + \frac{q^{3b_1/2-b_2/2+1}\left(1+q-2q^3\right)}{(1+2q)^{(b_1-b_2)/2+1}}, \tag{31}$$

and the incentive constraint (30) approaches, for $b_2 > b_1 > 0$,

$$q^2 \geq \frac{q^{b_1+1}}{1+q-q^2} + \frac{q^{3b_2/2-b_1/2+3}\left(1-q^2\right)}{(1+2q)^{(b_2-b_1)/2+1}\left(1+q-q^2\right)}, \tag{32}$$

and, finally, for $b_1 = b_2 = b > 0$,

$$q^2 \geq \frac{q\left(1+q+q^2\right)}{1+2q}q^b, \tag{33}$$

as $\delta$ goes to 1, where $b_1 - b_2$ and $b_2 - b_1$ must be even on the even integer lattice $\mathcal{B}$. Starting with the last, (33) is equivalent to

$$1 + 2q \geq q^{b-1}\left(1+q+q^2\right) \Leftrightarrow q \leq 1,$$

which therefore is true. Setting $m = \min\{b_1, b_2\}$ and $d = \max\{b_1, b_2\} - \min\{b_1, b_2\}$ the limiting incentive constraint (32) is equivalent to

$$(1+2q)^{d/2+1}\left(1+q-q^2-q^{m-1}\right) \geq \left(1-q^2\right)q^{3d/2+m+1},$$

and (31) is equivalent to

$$(1+2q)^{d/2+1}q^2(1-q^m) \geq \left(1+q-2q^3\right)q^{d/2+m+1}.$$

For the last two inequalities the left hand side is strictly increasing in $m$ and the rights hand side is strictly decreasing in $m$. Hence, both will hold for all

$m > 0$ if they do hold for $m = 1$, which is equivalent to

$$(1+2q)^{d/2+1} \geq (1+q)\, q^{3d/2+1} \text{ and}$$
$$(1+2q)^{d/2+1}(1-q) \geq \left(1+q-2q^3\right) q^{d/2}.$$

For the last two inequalities the left hand side is strictly increasing in $d$ and the right hand side is strictly decreasing in $d$. Therefore, both will hold for all $d = 2, 4, ...$ if they do hold for $d = 2$, which is equivalent to

$$(1+2q)^2 \geq (1+q)\, q^4 \Leftrightarrow 1 + q\,(1+q)\left(4-q^3\right) \geq 0 \text{ and}$$
$$(1+2q)^2(1-q) \geq \left(1+q-2q^3\right) q \Leftrightarrow 1 + 3q + 2q^2(1-q) \geq 0.$$

Since both are clearly fulfilled for any $q \in (0,1)$, the limiting incentive constraints do hold. By continuity the incentive constraints also hold for any $\delta \in (0,1)$ sufficiently large.

It remains to verify that at the boundary ($b_1 b_2 = 0$) there is no incentive to give (conditional on being successful), that is,

$$v^A(b_1+1) \geq v^A(b_1+1) \text{ and } w(1, b_2-1) \geq (1-\delta)\, v^A(0),$$

for $b_1 \geq 0$ and $b_2 > 0$. The first inequality follows from (3); the second from $w(1, b_2-1) \geq (1-\delta)\, v^A(2) > (1-\delta)\, v^A(0)$ also by (3). This complete the proof. □

**Proposition 5** For $p > 1/2$ and $\delta$ sufficiently large there is $M > 0$ such that there is a SPE with wealth-based sharing at all inventory states $(b_1, b_2) \in \mathcal{B}$ that satisfy $(b_1, b_2) \gg 0$ and

$$\max\{b_1, b_2\} \leq \min\{b_1, b_2\} + 2M,$$

and autarkic behavior outside this region.

*Proof.* Assume w.l.o.g. that individual $i = 2$ is at least as rich as $i = 1$, i.e. $0 \leq b_1 = b \leq b_2 = b + 2m$ for $m = 0, 1, ..., M$. Since the poor individual 1 behaves as under autarky, only the value for the rich individual 2 matters. Denote this normalized (multiplied by $1-\delta$) value by $v_2(b, b+2m)$. For $m = 0$ the statement follows from Proposition 4, because at the principal diagonal there is no difference between simple and wealth-based sharing. Therefore, consider the case $m = 1, 2, ..., M$. The value function $v_2$ satisfies, for $b > 0$ (in the even integer lattice),

$$\begin{aligned} v_2(b, b+2m) &= 1-\delta+\delta p^2 v_2(b+2, b+2+2(m-1)) \\ +\delta p(1-p) v_2 &+ \delta(1-p)^2 v_2(b-1, b-1+2m) \\ &+ \delta p(1-p) v_2(b+1, b+1+2(m-1)), \end{aligned} \tag{34}$$

and for $b=0$,

$$\begin{aligned} v_2(0, 2m) &= 1-\delta+\delta p^2 v_2(1, 1+2m)+\delta p(1-p)(1-\delta) v^A(2m+1) \\ &+\delta p(1-p) v_2(1, 1+2(m-1))+\delta(1-p)^2(1-\delta) v^A(2m-1), \end{aligned} \tag{35}$$

for all $m=1, ..., M$. At the principal diagonal, where $m=0$, the value is given by (27) with $b_2=b \geq 0$, that is,

$$v_2(b, b)=1-\frac{(1-p)(p+\mu/\delta)}{1-\delta p^2 \lambda} \lambda^b,$$

using (28). Proceeding by induction, assume that

$$v_2(b, b+2(m-1)) \geq(1-\delta) v^A(b+2(m-1)+1)$$

holds for all $b=1,2, \ldots$ and that $v_2(b-1, b-1+2m) \geq(1-\delta) v^A(b+2m)$ holds for some $b>1$. Then, from (34)

$$\begin{aligned} v_2(b, b+2m)= & \frac{1-\delta+\delta p^2 v_2(b+2, b+2+2(m-1))}{1-\delta p(1-p)} \\ & +\frac{\delta p(1-p) v_2(b+1, b+1+2(m-1))}{1-\delta p(1-p)} \\ & +\frac{\delta(1-p)^2 v_2(b-1, b-1+2m)}{1-\delta p(1-p)} \\ & \geq \frac{1-\delta+\delta p^2(1-\delta) v^A(b+1+2m)}{1-\delta p(1-p)} \\ & +\frac{\delta p(1-p)(1-\delta) v^A(b+2m)}{1-\delta p(1-p)} \\ & +\frac{\delta(1-p)^2(1-\delta) v^A(b+2m)}{1-\delta p(1-p)} \\ & =1-\frac{p^2 \mu^{b+2+2m}+(1-p) \mu^{b+1+2m}}{1-\delta p(1-p)} \end{aligned}$$

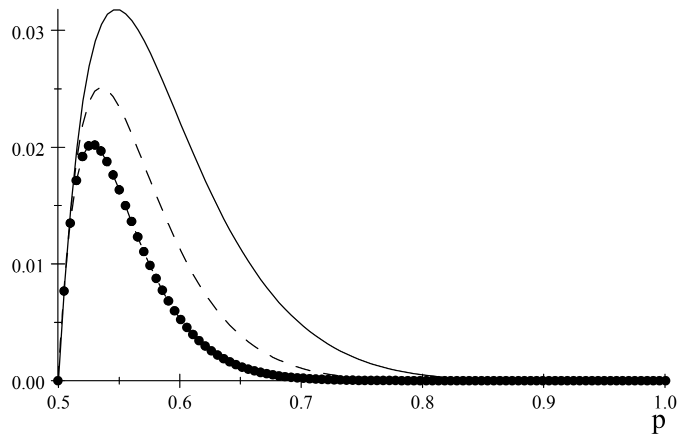


Figure 4: $\lim_{\delta\nearrow 1} v_2\left(1,1+2m\right)-\lim_{\delta\nearrow 1}\left(1-\delta\right)v^A\left(2+2m\right)$ for $m=1$ (solid). $m=2$ (dashed), and $m=3$ (dotted).

where the last expression approaches

$$\lim_{\delta\nearrow 1} v_2\left(b,b+2m\right)=1-\frac{1-p^2}{1-p\left(1-p\right)}\left(\frac{1-p}{p}\right)^{b+1+2m}$$

by (3) and (5). Therefore, we conclude from the fact that

$$1-\frac{1-p^2}{1-p\left(1-p\right)}\left(\frac{1-p}{p}\right)^{b+1+2m}>\lim_{\delta\nearrow 1}\left(1-\delta\right)v^A\left(b+2m\right)=1-\left(\frac{1-p}{p}\right)^{b+1+2m}$$

holds if and only if $p>1/2$ that the incentive constraints hold also at $b>1$ and $m$, for all $p>1/2$.

The difficult part of the argument is to show that

$$v_2\left(1,1+2m\right)\geq\left(1-\delta\right)v^A\left(2+2m\right).$$

Since induction over $m$ does not work for this case, we use an explicit solution to (34) and (35). In particular, defining

$$\varphi=\frac{\delta\left(1-p\right)^2}{1-\delta p\left(1-p\right)},\ \xi=\frac{\delta p\left(1-p\right)}{\left(1-\delta p\left(1-p\right)\right)^2},\ \text{and}\ \psi=\xi\varphi=\frac{\delta^2 p\left(1-p\right)^2}{\left(1-\delta p\left(1-p\right)\right)^3},$$

the bounded solution to (34) and (35) satisfies, for all $m = 0, 1, ..., M$ and all $b = 1, 2, ...,$

$$v_2(b, b+2m) = w(b+m, b+m) - \varphi^b \sum_{h=1}^{m} \psi^{m-h} C_m^h(b) \left[w(h,h) - v_2(0,2h)\right] \tag{36}$$

where $w(b,b)$ is given by (27) and the coefficients $C_m^h(b)$ satisfy $C_1^1(b) = C_m^m(b) = 1$ for all $b = 1, 2, ...,$ and

$$C_m^h(b) = \delta p(1-p) \sum_{k=3}^{b+2} C_{m-1}^h(k) + (1 - \delta p(1-p)) \sum_{k=2}^{b+1} C_{m-1}^h(k) \tag{37}$$

for all $h = 1, ..., m-1$, all $b = 1, 2, ...,$ and all $m = 1, ..., M$. That (36) solves (34) can be verified by direct substitution. The coefficients $C_m^h(b)$ can be computed recursively. They are given by polynomials in $b$ of order $m - h$ (without a constant term). To complete the recursive specification, the initial condition (35), given by the solution to the linear equation system (35) and (34) for $b = 1$, has to be determined. This yields

$$\begin{aligned} v_2(0,2m) = & \frac{1 - 2\delta p(1-p) - \delta^2 p^3 - (1-p)(1 - \delta p(1-p))\mu^{2m+1}/\delta}{1 - \delta p(1-p) - \delta^2 p^2 (1-p)^2} \\ & + \frac{\delta p(1-p)(1 - \delta p(1-p))}{1 - \delta p(1-p) - \delta^2 p^2 (1-p)^2} v_2(1, 1 + 2(m-1)) \\ & + \frac{\delta^2 p^3 (1-p)}{1 - \delta p(1-p) - \delta^2 p^2 (1-p)^2} v_2(2, 2 + 2(m-1)) \\ & + \frac{\delta^2 p^4}{1 - \delta p(1-p) - \delta^2 p^2 (1-p)^2} v_2(3, 3 + 2(m-1)). \end{aligned}$$

Substituting (36) into this equation and using (3), (27), (37), that

$$\lambda = \delta p^2 \lambda^2 + 2\delta p(1-p)\lambda + \delta(1-p)^2 \text{ and} \tag{38}$$

$$\mu = \delta p \mu^2 + \delta(1-p), \tag{39}$$

and the definitions of $\varphi$ and $\psi$, we obtain for all $m = 1, 2, ..., M$ that

$$\begin{aligned} v_2(0, 2m) = 1 &- \frac{(1-p)(1-\delta p(1-p))\mu^{2m+1}/\delta}{1-\delta p(1-p)-\delta^2 p^2 (1-p)^2} \\ &- \frac{\delta p^2 \lambda^2 + \delta^2 p (1-p)^3}{1-\delta p(1-p)-\delta^2 p^2 (1-p)^2} [1 - w(0,0)] \lambda^{m-1} \\ &- (1-p) \sum_{h=1}^{m-1} \psi^{m-h} D_m^h [w(h,h) - v_2(0, 2h)], \end{aligned}$$

where the coefficients $D_m^h$ are given by

$$\begin{aligned} D_m^h &= \frac{(1-\delta p(1-p))^3 C_{m-1}^h(1) + \delta^2 p^2 (1-p)^2 C_m^h(1)}{1-\delta p(1-p)-\delta^2 p^2 (1-p)^2} \text{ with} \\ C_m^h(1) &= \delta p (1-p) C_{m-1}^h(3) + (1-\delta p(1-p)) C_{m-1}^h(2) \end{aligned}$$

from (37) for $b = 1$. With this formulation the values $v_2(1, 1+2m)$ can be computed recursively starting with $m = 1$ and $b = 1, 2, ...$, then for $m = 2$ and $b = 1, 2, ...$, and so on. These computations become increasingly messy, though—hence the cautious formulation of the statement. For $m = 1, 2, 3$ the results of these computations are plotted in Figure 4 (for $p > 1/2$), which depicts the difference between $\lim_{\delta \nearrow 1} v_2(1, 1+2m)$ and $\lim_{\delta \nearrow 1} (1-\delta) v^A(2+2m)$ for $m = 1, 2, 3$. All those differences turn out to be positive, as required. □

**Proposition 6** For $p > 1/2$ and $\delta$ sufficiently large any strategy rule with ex-ante transfers that guarantees survival of both individuals everywhere except at the origin $(b_1, b_2) = (0, 0)$ constitutes a SPE for the stock game $\mathbf{G_s}$.

*Proof.* Denote the aggregate inventory by $B = b_1 + b_2$. A strategy rule that guarantees survival to both individuals except at the origin yields to either individual a normalized (multiplied by $1-\delta$) value function $\bar{v}^S(B)$ that satisfies

$$\bar{v}^S(B) = 1 - \delta + \delta p^2 \bar{v}^S(B+2) + 2\delta p(1-p)\bar{v}^S(B) + \delta(1-p)^2 \bar{v}^S(B-2) \quad (40)$$

for all $B = 2, 4, ...$ and

$$\bar{v}^S(0) = (1-\delta)p + \delta p^2 \bar{v}^S(2) + \delta p(1-p)(1-\delta) v^A(1) \quad (41)$$

for $B = 0$. The bounded solution to this linear second-order difference equation is given by $\bar{v}^S(B) = w(B/2, B/2)$, where the function $w(b_1, b_2)$ is given by (27) and (28), that is,

$$\bar{v}^S(B) = 1 - \frac{(1-p)\left(1+p\mu^2\right)}{1-\delta p^2\lambda}\lambda^{B/2} \tag{42}$$

(see (26) for the unnormalized version). That (42) solves (40) with initial condition (41) can be verified by direct substitution.

To see that guaranteed survival except at the origin constitutes equilibrium behavior, consider the incentive constraint at any inventory state $(b_1, b_2) \neq (0,0)$. Since the aggregate stock $B$ remains unchanged by ex-ante transfers, it becomes

$$\begin{aligned}
&\bar{v}^S(B) \geq (1-\delta)\, v^A(b_i) \Leftrightarrow \\
&1 - \frac{(1-p)\left(1+p\mu^2\right)}{1-\delta p^2\lambda}\lambda^{B/2} \geq 1 - \frac{\mu^{b_i+1}}{\delta} \Leftrightarrow \\
&\frac{\mu^{b_i+1}}{\delta} \geq \frac{(1-p)\left(1+p\mu^2\right)}{1-\delta p^2\lambda}\lambda^{B/2}
\end{aligned}$$

for $i = 1, 2$. As $\delta$ goes to 1 the last inequality approaches

$$\begin{aligned}
\left(\frac{1-p}{p}\right)^{b_i+1} &\geq \frac{1-p(1-p)}{1-(1-p)^2}\left(\frac{1-p}{p}\right)^{b_1+b_2+1} \Leftrightarrow \\
1 &\geq \frac{1-p(1-p)}{1-(1-p)^2}\left(\frac{1-p}{p}\right)^{b_{3-i}}
\end{aligned}$$

for $i = 1, 2$ and $p > 1/2$ by (5) and (12). Since $1-(1-p)^2 > 1-p(1-p) \Leftrightarrow p > 1/2$, both factors on the right hand side are smaller than 1, hence, so is their product—which verifies the incentive constraints. □