Implied Volatility Modelling

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Introduction

Black-Scholes Assumptions

- continuous trading
- constant interest rates with flat term structure
- it is possible to buy/short any number of asset or bond
- (from now on we assume also that dividend are 0)

The asset dynamics is assumed to follow geometric Brownian motion (GBM)

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

where
W_t is standard Wiener process
r interest rate
σ constant volatility
S_t asset price

Black-Scholes formula

The price of the European call option is given with:

$$C_t^{BS} = S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\ ,$$

where $d_{1,2} = \frac{\ln(S_t/K)+(r\pm\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$. $\Phi(u)$ is the CDF of the standard normal distribution, $\tau = T - t$ time to maturity, K the strike price.

Binomial trees

Binomial trees are discrete approximation of the GBM
They are powerful tool for pricing options

$$\Delta t = \frac{T}{n} \quad \text{with} \quad t_0 = 0,\ t_1 = \Delta t,\ t_2 = 2\Delta t,\ \ldots,\ t_n = n\Delta t = T$$

Changes in two directions:

$$\mathrm{P}(\text{up movement is } u) = p$$

$$\mathrm{P}(\text{down movement is } d) = q$$

Choose p, q and movements u, d such that for $\Delta t \longrightarrow 0$ and $n \longrightarrow \infty$, S_{j+1} will follow a geometric Brownian motion:

$$
\begin{aligned}
\ln S_{j+1} & \quad \text{normally distributed} \\
\mathrm{E}(\ln S_{j+1}) &= \ln S_j + \left(r - \frac{\sigma^2}{2}\right)\Delta t \\
Var(\ln S_{j+1}) &= \sigma^2 \Delta t
\end{aligned}
$$

We obtain

$$\begin{aligned} p + q &= 1, \\ \mathsf{E} \stackrel{\text{def}}{=} p \ln(uS_j) + q \ln(dS_j) &= \ln S_j + \left(r - \frac{1}{2}\sigma^2\right)\Delta t, \\ p\{\ln(uS_j) - \mathsf{E}\}^2 + q\{\ln(dS_j) - \mathsf{E}\}^2 &= \sigma^2 \Delta t. \end{aligned}$$

Put $q = 1 - p$, and we have:

$$\begin{aligned} p \ln\left(\frac{u}{d}\right) + \ln d &= \left(r - \frac{1}{2}\sigma^2\right)\Delta t, \\ p(1-p)\left\{\ln\left(\frac{u}{d}\right)\right\}^2 &= \sigma^2 \Delta t. \end{aligned}$$

This equation is nonlinear. Introducing a new condition $u \cdot d = 1$, the recombination property.
The recombination property yields $m_j = j + 1$ and

$$S_j^k = S_0 u^k d^{j-k}, \ \ k = 0, \cdots, j$$

We obtain:

$$p = \frac{1}{2} + \frac{1}{2}\left(r - \frac{1}{2}\sigma^2\right)\frac{\sqrt{\Delta t}}{\sigma} \tag{1}$$

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}$$

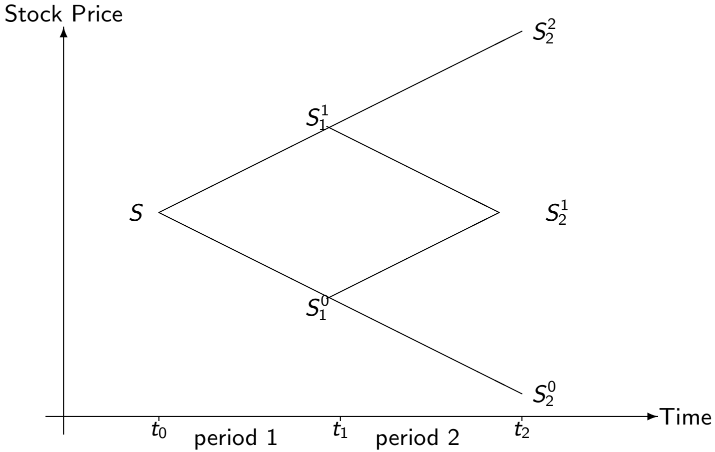


Figure 1: The generalized two period binomial model with the underlying asset values at the nodes.

Example: Call option without dividends

stock price S_t	230.00
exercise price K	210.00
time of expiration τ	0.50
volatility σ	0.25
risk free interest rate r	0.04545
dividend	no
steps	5
Call/Put	European call

Table 1: data for example (no dividend)

1. Up movement u: for $\Delta t = \tau/n = 0.1$, $u = 1.0823$ from (1)
2. Stock price S_n^k: from $S_0 = 230$ we calculate
   $S_1^1 = uS_0 = 248.92$ or $S_1^0 = S_0/u = 212.52, \cdots,$
   $S_5^5 = u^5 S_0 = 341.51$, $S_5^4 = u^3 S_0 = 291.56, \cdots,$
   $S_5^0 = S_0/u^5 = 154.90$
3. Option price at expiry date: $V_n^k = \max(0, S_n^k - K)$: e.g.
   $V_5^4 = V(S_5^4, t_5) = S_5^4 - K = 81.561$
4. Probability p: from (1), we get $p = 0.50898$.
5. Calculate option prices V_j^k

option price					
					341.51
				315.55	
			291.56		291.56
		269.40		269.40	
	248.92		248.92		248.92
230.00		230.00		230.00	
	212.52		212.52		212.52
		196.36		196.36	
			181.44		181.44
				167.65	
					154.90
0.00	0.10	0.20	0.30	0.40	0.50

Table 2: Stock price tree (no dividend)

stock price	option price					
341.50558						131.506
315.54682					106.497	
291.56126				83.457		81.561
269.39890			62.237		60.349	
248.92117		44.328		40.818		38.921
230.00000	30.378		26.175		20.951	
212.51708		16.200		11.238		2.517
196.36309			6.010		1.275	
181.43700				0.646		0.000
167.64549					0.000	
154.90230						0.000
time	0.00	0.10	0.20	0.30	0.40	0.50

Table 3: Option prices (no dividend)

Black-Scholes PDE

The price of the options can be obtained also by solving PDE:

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 C(S,t) S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS \frac{\partial C(S,t)}{\partial S} = rC(S,t) \quad (2)$$

Option Markets

- In modern financial markets vanilla options are regularly traded on the exchange
- One may trade simultaneously several options with different strike price and different maturity
- The unknown volatility parameter can be obtained from the market

Implied volatilities

Volatility $\hat{\sigma}$ as *implied* by observed market prices $\tilde{C}_t$:

$$\hat{\sigma}: \quad \tilde{C}_t - C_t^{BS}(S_t, K, \tau, r, \hat{\sigma}) = 0\ .$$

Implied Volatility Surface

The surface (on day t) given by the mapping from strikes and from time to maturity τ:

$$(K, \tau) \rightarrow \widehat{\sigma}_t(K, \tau)$$

is called *implied volatility surface* (IVS).

Moneyness

A convenient way of presenting the IVS is to rewrite it as a function of a moneyness and time to maturity. The moneyness κ is generally defined as:

$$\kappa = m(t, T, S_t, K, r).$$

where m is the increasing function in K.
From now on we will consider a IVS as function of moneyness κ and time to maturity τ:

$$(\kappa, \tau) \rightarrow \widehat{\sigma}_t(\kappa, \tau).$$

Moneyness

The moneyness can be defined in many different ways:

$$\begin{aligned}
\kappa_1 &= \frac{K}{S_t} \\
\kappa_2 &= \frac{K}{S_t e^{r\tau}} \\
\kappa_3 &= \ln\left(\frac{K}{S_t e^{r\tau}}\right) \\
\kappa_4 &= \frac{\ln\left(\frac{K}{S_t e^{r\tau}}\right)}{\sqrt{\tau}}
\end{aligned}$$

Data Design

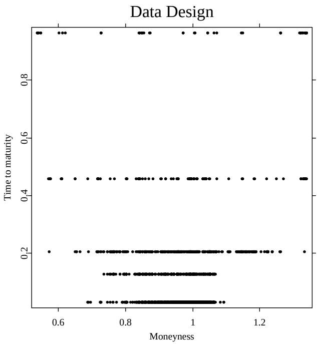


Figure 2: Data design on January, 4th 1999

Data Design

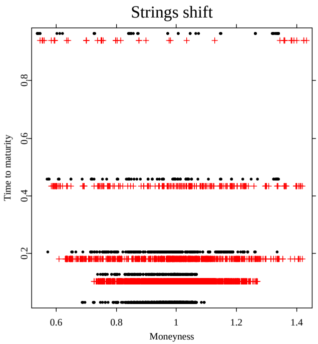


Figure 3: IV strings on January, 4th 1999 (points) and on January, 13th 1999 (crosses).

Smile

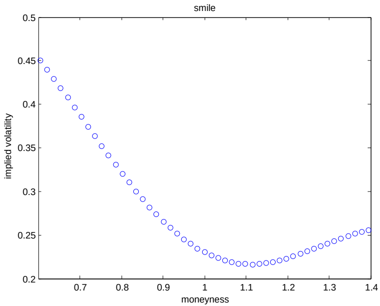


Figure 4: *An example of the smile*

Term structure

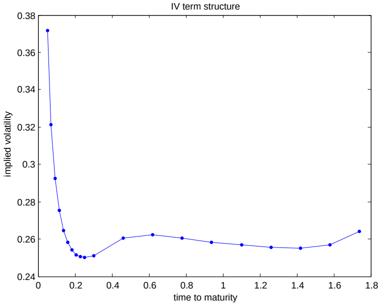


Figure 5: *An example of the term structure*

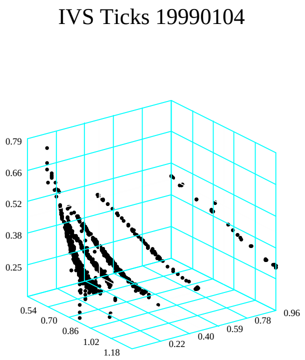


Figure 6: *An example of the IV ticks*

IVS Dynamics

- The IV strings move in space
- The IVS shifts in time randomly
- The IVS is subjected to the random deformations

Local Volatility Models

Local Volatility Model

The Black-Scholes model cannot replicate the observed vanilla option prices. Hence it is questionable to apply it for:

- hedging
- risk management
- pricing exotic options

Local Volatility Model

Dupire(1994) proposed the model which match observed option prices. The risk neutral process is set as:

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(S,t)dW_t$$

where:
$r(t)$ is an instantaneous interest rate
$\sigma(S,t)$ is an deterministic function of the spot and time.

Local Volatility Model

For the local volatility (LV) model one can derive the generalized BS PDE:

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V(S,t)}{\partial S^2} + r(t)S\frac{\partial V(S,t)}{\partial S} = r(t)V(S,t)$$

where $V(s,t)$ is a contingent claim on the asset S

Dupire Formula

The link between the local volatility surface (LVS) $\sigma(S_t, t)$ and the IVS is given by Dupire formula:

$$\sigma^2(S_t, t) = 2\frac{\frac{\partial C_t(K,T)}{\partial T} + rK\frac{\partial C_t(K,T)}{\partial K}}{K^2\frac{\partial^2 C_t(K,T)}{\partial K^2}}$$

Overview

1. introduction and definitions✓
2. implied binomial trees
3. Andersen and Brotherthon-Ratcliffe finite difference method

IBT Algorithm

Notations and Assumptions

- $s_{n,i}$, the stock price of the ith node at the n-th level
- Forward prices $F_{n,i} = s_{n,i} \times e^{\triangle t}$ and transition probabilities $p_{n,i}$ satisfy the risk-neutral condition:

$$F_{n,i} = p_{n,i} s_{n+1,i+1} + (1 - p_{n,i}) s_{n+1,i}$$

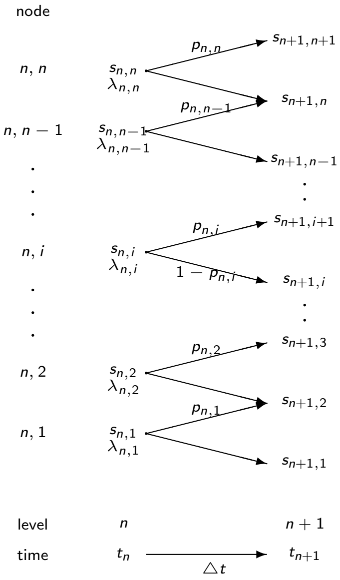
- $F_{n,i} < s_{n+1,i+1} < F_{n,i+1}$, in order to avoid arbitrage.
- Arrow-Debreu prices $\lambda_{n,i}$ (discounted risk-neutral probability) the price of an option that pays 1 in one and only one state i at nth level, and otherwise pays 0.

$$
\begin{aligned}
\lambda_{n+1,1} &= e^{-r\triangle t}\left\{(1-p_{n,1})\lambda_{n,1}\right\} \\
\lambda_{n+1,i+1} &= e^{-r\triangle t}\left\{\lambda_{n,i}p_{n,i}+\lambda_{n,i+1}(1-p_{n,i+1})\right\}, \quad 2\le i\le n \\
\lambda_{n+1,n+1} &= e^{-r\triangle t}\left\{\lambda_{n,n}p_{n,n}\right\}
\end{aligned}
$$

$\triangle t$, length of the time interval.

- Call option price $C(K,n\triangle t)$ and put option price $P(K,n\triangle t)$

$$
\begin{aligned}
C(K,n\triangle t) &= \sum_{i=1}^{n+1}\lambda_{n+1,i}\max(s_{n+1,i}-K,0) \\
P(K,n\triangle t) &= \sum_{i=1}^{n+1}\lambda_{n+1,i}\max(K-s_{n+1,i},0)
\end{aligned}
$$


node
n, n
n, n − 1
n, i
n, 2
n, 1
s_{n,n}
λ_{n,n}
s_{n,n−1}
λ_{n,n−1}
s_{n,i}
λ_{n,i}
s_{n,2}
λ_{n,2}
s_{n,1}
λ_{n,1}
p_{n,n}
p_{n,n−1}
p_{n,i}
1 − p_{n,i}
p_{n,2}
p_{n,1}
s_{n+1,n+1}
s_{n+1,n}
s_{n+1,n−1}
s_{n+1,i+1}
s_{n+1,i}
s_{n+1,3}
s_{n+1,2}
s_{n+1,1}
level
n
n + 1
time
t_n
t_{n+1}
△t

Derman and Kani (D & K) IBT

Step 1: Central nodes

- Define $s_{n+1,i} = s_{1,1} = S$, $i = n/2 + 1$, , for n even
- Start from $s_{n+1,i}$, $s_{n+1,i+1}$, $i = (n+1)/2$,
  suppose $s_{n+1,i} = s_{n,i}^2 / s_{n+1,i+1} = S^2 / s_{n+1,i+1}$, for n odd

$$s_{n+1,i+1} = \frac{S\{e^{r\triangle t} C(S, n\triangle t) + \lambda_{n,i} S - \rho_u\}}{\lambda_{n,i} F_{n,i} - e^{r\triangle t} C(S, n\triangle t) + \rho_u} \quad \text{for} \quad i = (n+1)/2$$

Step 2: Upward

$$s_{n+1,i+1} = \frac{s_{n,i}\{e^{r\triangle t}C(s_{n,i}, n\triangle t) - \rho_u\} - \lambda_{n,i}s_{n,i}(F_{n,i} - s_{n+1,i})}{\{e^{r\triangle t}C(s_{n,i}, n\triangle t) - \rho_u\} - \lambda_{n,i}(F_{n,i} - s_{n+1,i})}$$

Step 3: Downward

$$s_{n+1,i} = \frac{s_{n,i+1}\{e^{r\triangle t}P(s_{n,i}, n\triangle t) - \rho_l\} - \lambda_{n,i}s_{n,i}(F_{n,i} - s_{n+1,i+1})}{\{e^{r\triangle t}P(s_{n,i}, n\triangle t) - \rho_l\} + \lambda_{n,i}(F_{n,i} - s_{n+1,i+1})}$$

where

$$\rho_u = \sum_{j=i+1}^{n} \lambda_{n,j}(F_{n,j} - s_{n,i}), \rho_l = \sum_{j=1}^{i-1} \lambda_{n,j}(s_{n,i} - F_{n,j})$$

Technical Summary for D & K construction:

1. Start from the central nodes, calculate their stock prices
2. Calculate the transition probabilities and Arrow-Debreu prices at the corresponding nodes
3. Calculate stock prices upward or downward, where interpolated option prices are estimated by CRR method
4. Repeat 2 and 3

Example

$S = 100$, $r = 3\%$, the annual BS implied volatility of a call is $\sigma = 10\%$, the BS implied volatility $\sigma_{imp}(K, \tau) = 0.15 - 0.0005K$.

Output four-step four-year IBT of stock prices, transition probabilities, Arrow-Debreu prices respectively:

D & K IBT

$T = 1$ year, $\triangle t = 1/4$ year

stock price

				119.91
			115.07	
		110.05		110.06
	105.13		105.14	
100.00		100.00		100.00
	95.12		95.11	
		89.93		89.93
			85.21	
				80.02

D & K IBT

$T = 1$ year, $\triangle t = 1/4$ year

transition probability

			0.596
		0.578	
	0.589		0.590
0.563		0.563	
	0.587		0.586
		0.545	
			0.589

D & K IBT

$T = 1$ year, $\triangle t = 1/4$ year

Arrow-Debreu price

				0.111
			0.187	
		0.327		0.312
	0.559		0.405	
1.000		0.480		0.342
	0.434		0.305	
		0.178		0.172
			0.080	
				0.033

Barle and Cakici (B & C) IBT

Major modifications

- Align the central nodes of the tree with the forward price rather than with the current stock price
- Use the forward price of the previous node to calculate the new option of the nodes at the next level
- Use Black-Scholes formula instead of CRR binomial tree method to calculate the interpolated option prices

Step 1: Central nodes

- Define $s_{n+1,i} = s_{1,1} \cdot e^{rn\triangle t} = S \cdot e^{rn\triangle t}$, $i = n/2 + 1$, for n even
- Start from $s_{n+1,i}$, $s_{n+1,i+1}$, $i = (n+1)/2$, suppose $s_{n+1,i} = F^2_{n,i}/s_{n+1,i+1}$, for n odd

$$s_{n+1,i} = F_{n,i} \frac{\lambda_{n,i} F_{n,i} - \triangle^C_{n,i}}{\lambda_{n,i} F_{n,i} + \triangle^C_{n,i}} \quad \text{for} \quad i = (n+1)/2$$

Step 2: Upward

$$s_{n+1,i+1} = \frac{\triangle^C_{n,i} s_{n+1,i} - \lambda_{n,i} F_{n,i}(F_{n,i} - s_{n+1,i})}{\triangle^C_{n,i} - \lambda_{n,i}(F_{n,i} - s_{n+1,i})}$$

Step 3: Downward

$$s_{n+1,i} = \frac{\lambda_{n,i} F_{n,i}(s_{n+1,i+1} - F_{n,i}) - \triangle^P_{n,i} s_{n+1,i+1}}{\lambda_{n,i}(s_{n+1,i+1} - F_{n,i}) - \triangle^P_{n,i}}$$

where

$$\begin{aligned}
\triangle^C_{n,i} &= e^{r\triangle t} C(F_{n,i}, n\triangle t) - \sum_{j=i+1}^{n} \lambda_{n,j}(F_{n,j} - F_{n,i}), \\
\triangle^P_{n,i} &= e^{r\triangle t} P(F_{n,i}, n\triangle t) - \sum_{j=1}^{i-1} \lambda_{n,j}(F_{n,i} - F_{n,j}).
\end{aligned}$$

B & C IBT

$T = 1$ year, $\triangle t = 1/4$ year

stock price

				123.85
			117.02	
		112.23		112.93
	104.84		107.03	
100.00		101.51		103.05
	96.83		97.73	
		90.53		93.08
			87.60	
				82.00

Given an implied tree, the local volatility $\sigma_{i,j}$ is calculated

$$
\begin{aligned}
\mu_{i,j} &= p_{i,j} R_{i+1,j+1} + (1 - p_{i,j}) R_{i,j+1} \\
\sigma^2_{i,j} &= p_{i,j} (R_{i+1,j+1} - \mu_{i,j})^2 + (1 - p_{i,j}) (R_{i,j+1} - \mu_{i,j})^2
\end{aligned}
$$

Finite-Difference Approach

- Practitioners employ finite-difference approach to calibrate the LVS and price exotic options Andersen and Brotherton-Ratcliffe (1998)
- The algorithm provides accurate and stable fit
- It yields fast pricing algorithms

Recall the PDE for pricing the contingent claim $V(S_t, t)$

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V(S,t)}{\partial S^2} + r(t)S\frac{\partial V(S,t)}{\partial S} = r(t)V(S,t) \tag{3}$$

with boundary condition $V(S_T, T) = g(S_T)$.

The instantaneous interest rate r is a deterministic function of time

$$r(t) = \frac{-\partial P(0,t)/\partial t}{P(0,t)}$$

where $P(t,T) = \exp[-\int_t^T r(u)du]$ is the price of the zero-coupon bond.

For some technical reason in (3) put $x = lnS$ and $H(x,t) = V(S,t)$ replace the equation with

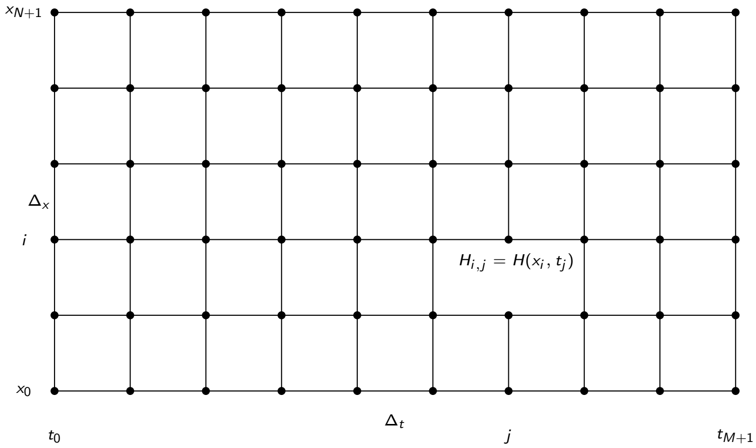
$$\frac{\partial H(x,t)}{\partial t} + \frac{1}{2} v(x,t) \frac{\partial^2 H(x,t)}{\partial x^2} + b(x,t) \frac{\partial H(x,t)}{\partial x} = r(t) H(x,t) \quad (4)$$

where $b(x,t) = r(t) - \frac{1}{2} v(x,t)$, $v(x,t) = \sigma^2(S,t) = \sigma(e^x,t)$. The functions have to be calibrated so that the discretzation will return correct prices market prices of the bonds and European options.

Discretization

Divide the (x, t) plane into a uniformly spaced mesh with $M+2$ nodes along the t axis and $N+2$ nodes along the x axes:

$$
\begin{aligned}
x_i &= x_0 + i\Delta_x = x_0 + i\frac{x_{N+1} - x_0}{N+1}, \quad i = 0, ..., N+1 \\
t_j &= j\Delta_t = j\frac{T}{M+1}, \quad j = 0, ..., M+1 \\
x_{ini} &= lnS_{ini} = x_\beta
\end{aligned}
\tag{5}
$$


x_{N+1}
Δ_x
i
x_0
$H_{i,j} = H(x_i, t_j)$
Δ_t
t_0
j
t_{M+1}

Discretization

$$
\begin{aligned}
\frac{\partial H}{\partial t} &\approx \frac{H(x_i, t_{j+1}) - H(x_i, t_j)}{\Delta_t} \\
\frac{\partial H}{\partial x} &\approx (1-\Theta)\frac{H(x_{i+1}, t_j) - H(x_{i-1}, t_j)}{2\Delta_x} + \Theta\frac{H(x_{i+1}, t_{j+1}) - H(x_{i-1}, t_{j+1})}{2\Delta_x} \\
\frac{\partial^2 H}{\partial x^2} &\approx (1-\Theta)\frac{H(x_{i+1}, t_j) - H(x_i, t_j) + H(x_{i-1}, t_j)}{\Delta_x^2} \\
&+ \Theta\frac{H(x_{i+1}, t_{j+1}) - H(x_i, t_{j+1}) + H(x_{i-1}, t_{j+1})}{\Delta_x^2}
\end{aligned}
\tag{6}
$$

- $\Theta = 0$ - fully implicit finite-difference method
- $\Theta = 1$ - explicit finite-difference method
- $\Theta = \frac{1}{2}$ - Crank-Nicholson scheme

Plugging discretization schemes to (3) leads to system of equations written in matrix notation:

$$[(1 + r_j \Delta_t)\mathbf{I} - (1 - \Theta)\mathbf{M}_j]\mathbf{H}_j = (\Theta \mathbf{M}_j + \mathbf{I})\mathbf{H}_{j+1} + \mathbf{B}_j \qquad (7)$$

for each $j = 0, ..., M$. $\mathbf{I}$ is the identity matrix

$$\mathbf{H}_j = \begin{pmatrix} H_{1,j} \\ H_{2,j} \\ \vdots \\ H_{N,j} \end{pmatrix}$$

$$\mathbf{B}_j = \begin{pmatrix} l_{1,j}[(1-\Theta)H_{0,j} + \Theta H_{0,j+1}] \\ 0 \\ \vdots \\ 0 \\ u_{N,j}[(1-\Theta)H_{N+1,j} + \Theta H_{N+1,j+1}] \end{pmatrix}$$

$$\mathbf{M}_j = \begin{pmatrix} c_{1,j} & u_{1,j} & 0 & 0 & 0 & \cdots & 0 \\ l_{2,j} & c_{2,j} & u_{2,j} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l_{N-1,j} & c_{N-1,j} & u_{N-1,j} \\ 0 & 0 & 0 & \cdots & 0 & l_{N,j} & c_{N,j} \end{pmatrix}$$

where

$$\begin{aligned} c_{i,j} &= -\alpha v_{i,j} \\ u_{i,j} &= \frac{1}{2}\alpha(v_{i,j} + \Delta_x b_{i,j} \\ l_{i,j} &= \frac{1}{2}\alpha(v_{i,j} - \Delta_x b_{i,j} \\ \alpha &= \Delta_t/\Delta_x^2 \end{aligned}$$

- If $r_j, v_{i,j}, b_{i,j}$ are known the only unknown the only unknown quantities are $H_{1,j}, ..., H_{N,j}$.
- Because from the payout at time T the vector $\mathbf{H}_{N+1}$ is known the solution of the whole system can be obtained iterative backward induction.
- Numerical solution can be coded very efficiently

Fitting of Bond Prices

The bonds prices are described by

$$P(T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}$$

For each time j the price is "given" on the market $P_j = P(0, t_j)$
Then

$$P(t_j, t_{j+1})(1 + r_j \Delta_t) = P(t_{j+1}, t_{j+1}) = 1$$

and

$$r_j = \frac{1}{\Delta_t}(\frac{P_j}{P_{j+1}} - 1)$$

Fitting European Call Options

- Assume the existence of observable call prices on the all nodes
- Denote $C_{ini}^{i,j}$ the value of a European call with strike $K = S_i = e^{x_i}$ and maturity t_i.
- Let $A_{ini}^{i,j}$ be an Arrow-Debreu security that pays 1 if at the time t_j the asset price is equal S_i and zero otherwise.

The Arrow-Debreu securities must satisfy

$$\sum_{i=0}^{N+1} A_{ini}^{i,j} = P_j$$

$$\sum_{i=0}^{N+1} A_{ini}^{i,j} S_i = S_{ini}$$

The call prices needs to satisfy

$$C_{ini}^{i,j} = \sum_{l=i+1}^{N+1} A_{ini}^{l,j}(S_l - S_i)$$

for $i = 1, ..., N$ and $j = 0, ..., M + 1$

For the lower and upper boundary

$$C_{ini}^{N+1,j} = 0$$

$$C_{ini}^{0,j} = \sum_{l=0}^{N+1} A_{ini}^{l,j} S_l - S_0 \sum_{l=0}^{N+1} A_{ini}^{i,j} = S_{ini} - S_0 P_j$$

for $j = 0, ..., M+1$.

After some calculations:

$$A_{ini}^{i,j} = \frac{(S_i - S_{i-1})C_{ini}^{i+1,j} - (S_{i+1} - S_i)C_{ini}^{i,j} + (S_{i+1} - S_i)C_{ini}^{i-1,j}}{(S_{i+1} - S_i)(S_i - S_{i-1})}$$
$$= \frac{e^{-\frac{1}{2}\Delta_x} C_{ini}^{i+1,j} - 2\cosh \frac{1}{2}\Delta_x C_{ini}^{i,j} + e^{\frac{1}{2}\Delta_x} C_{ini}^{i-1,j}}{2e^{x_i} \sinh \frac{1}{2}\Delta_x}$$

for $i = 1, ..., N$ and $j = 1, ..., M+1$
For the upper boundary

$$A_{ini}^{i,j} = \frac{C_{ini}^{N,j}}{(e^{\Delta_x} - 1)e^{x_N}}$$

For the lower boundary

$$A_{ini}^{i,j} = \frac{P_j e^{\Delta_x} - e^{\Delta_x} + C_{ini}^{1,j}/e^{x_0}}{e^{\Delta_x} - 1}$$

For the $j = 0$ $A_{ini}^{i,0} = 1$ if $i = \beta$ and $A_{ini}^{i,0} = 0$ otherwise

Arrow-Debreu securities must satisfy (7). After some algebraic calculations one obtains:

$$(\mathbf{M}_j)^\top((1-\Theta)\mathbf{A}_{ini}^{j+1}+\Theta\mathbf{A}_{ini}^{j}) = \frac{P_j}{P_{j+1}}\mathbf{A}_{ini}^{j+1}-\mathbf{A}_{ini}^{j} \qquad (8)$$

where

$$\mathbf{A}_{ini}^{j} = \begin{pmatrix} A_{ini}^{1,j} \\ A_{ini}^{2,j} \\ \vdots \\ A_{ini}^{N,j} \end{pmatrix}$$

$\mathbf{M}_j$ contains the v_j.
Solving (8) wrt. v_j yields the local volatilities.

Merton Model and Levy Models

Black-Scholes model

In Black-Scholes model price of the asset is modelled with:

$$S_t = S_0 \exp(\sigma W_t + (r - \frac{\sigma^2}{2})t), \ \ t \geq 0$$

where:
S_t is the asset's price in time point t
S_0 is the asset's price in a current time point ($t = 0$)
r is an interest rate
W_t is a standard Wiener process
σ is a volatility

Option price in Black-Scholes model

In Black-Scholes model one can derive price for the call option as:

$$C(S_0, K, \tau, r) = S_0\Phi(y + \sigma\sqrt{\tau}) - e^{-r\tau}\Phi(y)$$

where

$$y = \frac{\log \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$\Phi(\cdot)$ is standard normal distribution function.

Put-call parity of European option: for a put and a call with the same expiry, the same strike price and based on the same underlying, it holds that

$$P = C - S_t + Ke^{-r(T-t)}$$

where C is the call price and P the put price at t, r is the risk-free interest rate.

Parameters of the option price:

- S_0 value of underlying asset
- K exercise price or strike price
- r interest rate
- τ time to maturity
- σ volatility

S_0 and r are values taken from the market.
K and τ are specified in the option contract.
σ is unknown parameter which measures incertainity of future changes of price. It is often estimated as standard deviation from returns:

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \frac{1}{n}\sum_{j=1}^{n} Z_j)^2}$$

Implied volatility

On the market the prices of derivatives are determined by the law of supply and demand. It means that $C(S_0, K, \tau, r)$ is observed. In Black-Scholes formula for call option price only σ is not obeserved

$$C(S_0, K, \tau, r) = S_0 \Phi(y + \sigma\sqrt{\tau}) - e^{-r\tau}\Phi(y)$$

$$y = \frac{\log \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

Implied volatility

Since the option price is a monotonic function of volatility it is possible to find unique parameter σ_I such that match the equation:

$$C^{BS}(S_0, K, \tau, r, \sigma_I) = C^*(K, \tau)$$

where
$C^{BS}(S_0, K, \tau, r, \sigma_I)$ is a price given with Black-Scholes formula
$C^*(K, \tau)$ is the price obcerved in the market
σ_I is called **implied volatility**.

Since the Black-Scholes formula is complicated the implied volatility is not given explicitly One needs to use numerical techniques to obtain the result.
Prices on the option market are commonly quoted in terms of Black-Scholes implied volatility. Black-Scholes formula is not used as a pricing model but as a tool for representing prices in terms of implied volatility.

In Black-Scholes model σ is assumed to be constant. In real markets implied volatilities exhibit non constant behavior.
If we denote the implied volatility by $\sigma_I(K, T)$ then the surface $\sigma_I(K, T)_{K,T}$ contains the implied volatility for all strikes and maturities.
On the option market we can observe only few point from this surface.

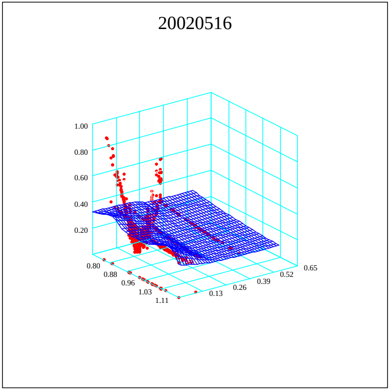


Figure 7: Implied volatility surface

Merton model

In Merton model the price of an asset is modelled as:

$$S_t = S_0 \exp\{\gamma t + \sigma W_t + \sum_{i \geq 1}^{N_t} Y_i\}$$

W_t is standard Wiener process
N_t is Poisson process with intensity λ independent from W_t
$Y_i \sim N(\mu, \delta^2)$ are i.i.d independent from W_t and N_t
$\gamma = r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)$

In Black-Scholes model generated implied volatility surface is constant what is in contradiction with observed option prices.
In Merton model generated implied volatility surface is not constant and replicate the behavior of option prices more realistic.

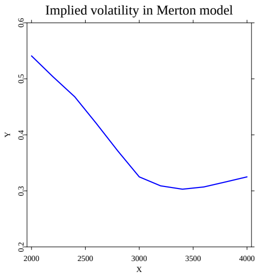


Figure 8: Implied volatilities genereted in Merton model
smilemerton.xpl

Calibration problem

In order to use Merton model efficiently (for risk managment or pricing exotic options) one needs to specify set of parameters $(\lambda, \sigma, \delta, \mu)$.

- λ intensity of jumps
- σ volatility
- δ standard deviation of jumps
- μ mean of jumps

The specyfing the set of parameters is called calibration of the model.

In calculating implied volatilities in Black-Scholes model one has one option and one parameter. The solution is unique.
In calibration of the Merton model there are more options and four parameters. The solution does not need to be unique.
The idea of calibration is to search for model parameters that minimize the distance between the IVS of the model and an IVS observed on the market.

Minimizing function

$$f(\Theta) = \sum_i \frac{(C_i^* - C_i^M(\Theta))^2}{C_i^*} \mathbf{1}_{S \leq K} + \sum_i \frac{(P_i^* - P_i^M(\Theta))^2}{P_i^*} \mathbf{1}_{S > K}$$

where:
Θ is the set of parameters $(\lambda, \sigma, \delta, \mu)$
C_i^*, P_i^* call/put option prices from the market
$C_i^M(\Theta), P_i^M(\Theta)$ option prices calculated with Merton model with parameters Θ

Estimated parameters of the Merton model:

- $\widehat{\lambda} = 0.950717$
- $\widehat{\sigma} = 0.100115$
- $\widehat{\delta} = 0.119883$
- $\widehat{\mu} = -0.109419$

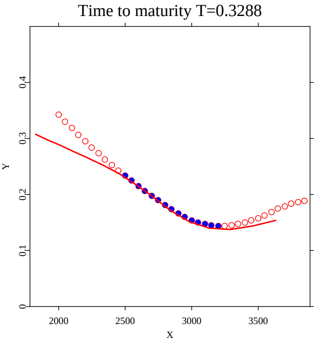


Figure 9: Implied volatility of calibrated Merton model. Blue points denote which options were taken into calibration

Problems with efficient calibration

There are several problems how to effiiently and accuratly calibrate parameters of the Merton

- Option pricing function needs to be called many times. Monte Carlo pricing function works too slow so using it in calibration is not reasonable.
- The parameters' space has four dimensions. It is hard to tell anything about minimizing function.

Fast method of option pricing

Carr, Madan proposed a method for option valuation based on the fast Fourier transform(FFT).
Some motivations for the use of FFT:

- the considerable power of the FFT
- the Fourier transform of the (logarithm of the) price process is known for many models specially models based on Lévy processes like Merton model
- FFT allows to calculate prices for a whole range of strikes

Pricing a single call

The value $C_T(k)$ of a T-maturity call with strike $K = \exp(k)$ is given by

$$C_T(k) = \int_k^\infty e^{-rT}(e^s - e^k) q_T(s) ds$$

where q_T is the density of the log price S_T.
As the function C_T is not square-integrable we cannot apply the Fourier inversion directly. Thus we consider the modified function

$$c_T(k) = \exp(\alpha k) C_T(k)$$

which should be square-integrable for a suitable $\alpha > 0$.

The Fourier transform of c_T is defined by

$$\psi(v) = \int_{-\infty}^{\infty} e^{\mathbf{i}vk} c_T(k) dk.$$

The Fourier transform ψ can be expressed as well:

$$\psi(v) = \frac{e^{-rT}\phi(v - (\alpha+1)\mathbf{i})}{\alpha^2 + \alpha - v^2 + \mathbf{i}(2\alpha+1)v}$$

where ϕ is the Fourier transform of q_T.

Example

In Merton model $\phi(u) = e^{-\frac{\sigma^2 u^2 t}{2} + i\gamma ut + \lambda t(e^{-\delta^2 u^2/2 + \mathbf{i}\mu u} - 1)}$ where:
$\gamma = r - \frac{\sigma^2}{2} - \lambda(e^{\mu+\delta^2/2} - 1)$

As c_T is square-integrable we can get back the call price by applying the inverse Fourier transform

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \int_0^\infty e^{-\mathbf{i}vk} \psi(v) dv.$$

The call price can be computed numerically using the trapezoid rule

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=0}^{N-1} w_j e^{-\mathbf{i}v_j k} \psi(v_j) \eta$$

where $v_j = \eta j,\ j = 0, \ldots, N-1$ with some $\eta > 0$.
$w_0 = w_{N-1} = \frac{1}{2}$ and $w_1 = ... = w_{N-2} = 1$

Pricing calls with different strikes

Let us consider now N calls with maturity T and strikes

$$k_u = -\frac{1}{2}N\lambda + \lambda u, \;\; u = 0, \ldots, N-1$$

where $\lambda > 0$ is the distance between the log strikes.
The formula for the numerical approximation of the call price gives

$$C_T(k_u) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=0}^{N-1} w_j e^{-\mathbf{i}\lambda\eta j u} e^{\mathbf{i}\frac{1}{2}N\lambda v_j} \psi(v_j)\eta, \;\; u = 0, \ldots, N-1.$$

This representation allows a direct application of the FFT which is an efficient algorithm for computing the sum

$$a_k = \sum_{j=0}^{N-1} e^{-\mathbf{i}\frac{2\pi}{N}jk} x_j, \ \ k = 0, \dots, N-1.$$

The parameters λ, η, N only have to satisfy the constraint

$$\lambda\eta = \frac{2\pi}{N}.$$

If we choose η small in order to obtain a fine grid for the numerical integration, then we observe call prices at relatively large strike spacings, with few strikes lying in the desired region near the stock price.

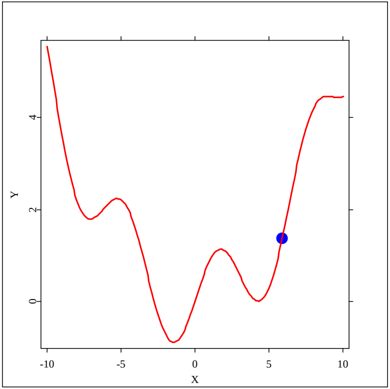
FFT versus MC

FFT time: 0.015 sec.
MC time: 36.531 sec. (5000 simulations, 500 time steps)
disadvantages of FFT

- instable for fixed FFT parameter α, η, N
- applicable only to european options

Searching for minimum

In order to find parameters of the model one needs to minimize numericaly appropriate function.
The minimizing function could have many local minimums what makes the problem more difficult.
The performance of the method can also depends on starting values of the algorithm. There is no rule how to set the starting point.

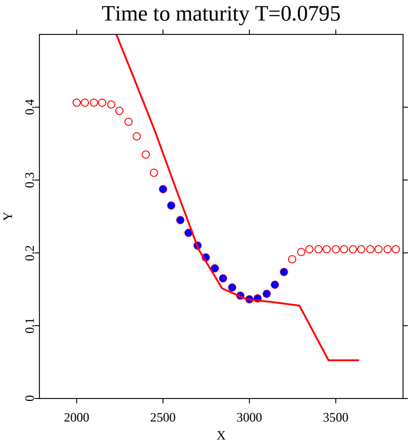

Y
0
2
4
-10
-5
0
5
10
X

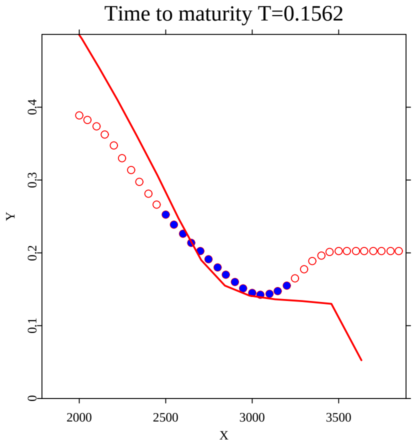
Simulated annealing

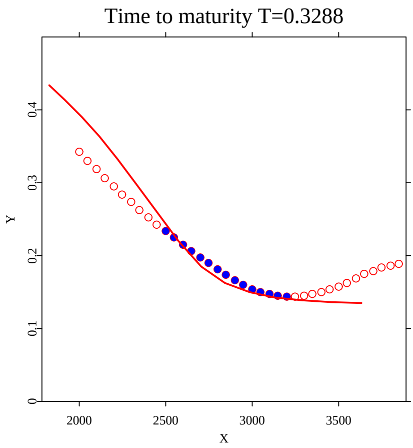
Simulated annealing is a numerical algorithm for finding a global minimum of a function. Each step of the algorithm is adjusted by adding some random variable with variance T. After certain amount of function calls T is decreased and algorthm is restarted from the best ever point. There is a hope that due to random adjustment the algorithm will jump out of the local minimium valey and find valey with global minimum.

Merton model is not good when we think about whole surface that is why even more comlicated models need to be consider.
Estimated parameters of the Merton model for six different maturietes:

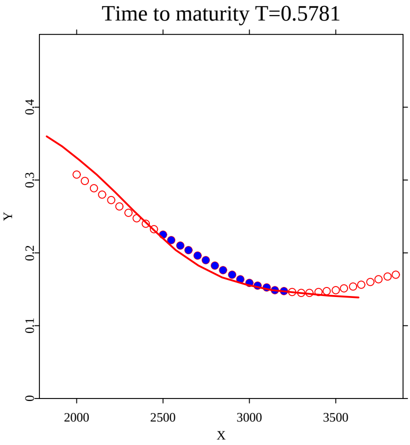
- $\widehat{\lambda} = 0.096349$
- $\widehat{\sigma} = 0.127587$
- $\widehat{\delta} = 0.17323$
- $\widehat{\mu} = -0.568271$

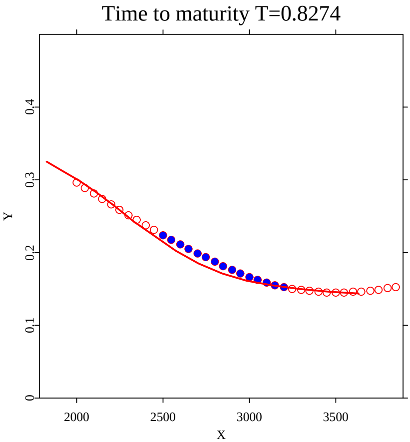

Time to maturity T=0.0795
0
0.1
0.2
0.3
0.4
Y
2000
2500
3000
3500
X

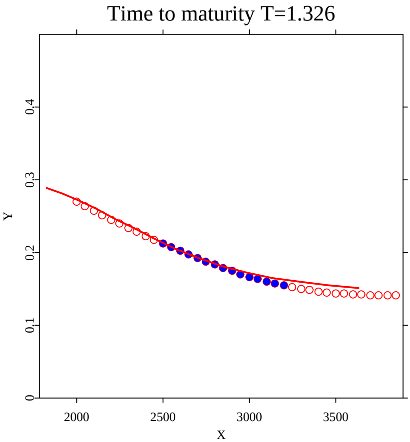

Time to maturity T=0.1562
Y
0
0.1
0.2
0.3
0.4
2000
2500
3000
3500
X



Time to maturity T=0.3288
Y
0
0.1
0.2
0.3
0.4
2000
2500
3000
3500
X


Time to maturity T=0.5781
Y
0
0.1
0.2
0.3
0.4
2000
2500
3000
3500
X


Time to maturity T=0.8274
Y
X
0
0.1
0.2
0.3
0.4
2000
2500
3000
3500


Time to maturity T=1.326
Y
X
0
0.1
0.2
0.3
0.4
2000
2500
3000
3500

Wiener process

(i) $W_0 = 0$

(ii) $W_t \sim N(0, t),\ \ t \geq 0$

(iii) $\{W_t;\ \ t \geq 0\}$ has independent increments: $W_t - W_s$ is independent from $W_s, \quad \forall\ t > s \geq 0$

(iv) $(W_t - W_s) \sim N(0, (t - s))$

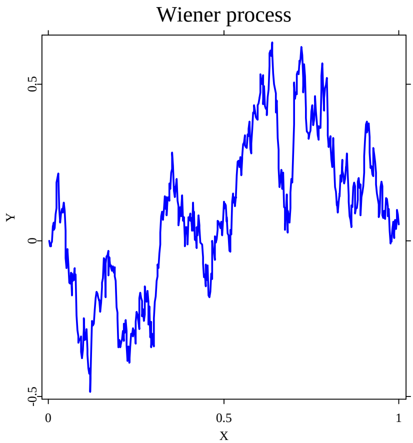


Figure 10: Typical paths of Wiener process genwiener.xpl

Poisson process

Exponential distribution is the distribution with density $\lambda e^{\lambda x}\mathbf{1}_{x\geq 0}$
Let τ_i be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$.
The process $(N_t, t \geq 0)$ defined by

$$N_t = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n}$$

is called Poisson process with intensity λ.

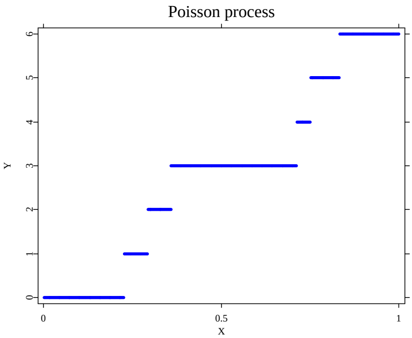


Figure 11: Typical paths of Poisson process genpoiss.xpl

Compound Poisson process

A compound Poisson process with intensity λ is a stochastic process X_t defined as:

$$X_t = \sum_{i \geq 1}^{N_t} Y_i$$

where:
Y_i are i.i.d. with distribution f and N_t is a Poisson process with intensity λ. N_t is independent from Y_i.
When $Y_i = 1$ we obtain standard Poisson process.

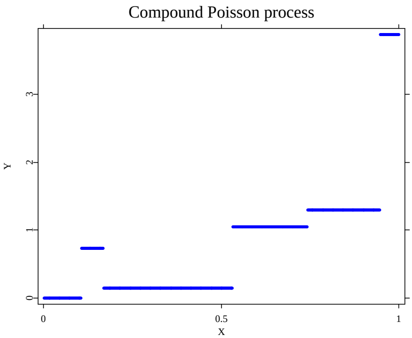


Figure 12: Typical paths of compound Poisson process with standard normal distribution of jump size gencpoiss.xpl

Simple Lévy process can be created from independent Brownian motion with drift and diffusion coefficient ($\gamma t + aW_t$) and compound Poisson process C_t

$$X_t = \gamma t + aW_t + C_t$$

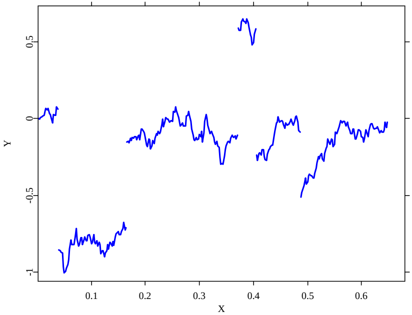


Figure 13: Typical paths of Lévy process composed from Brownian motion with drift and compound Poisson process genlevy.xpl

Lévy process

A stochastic process $(X_t)_{t \geq 0}$ in $\mathbb{R}$ is called Lévy process if :

(i) (X_t) has independent and stationary increments.

(ii) $X_0 = 0$

(iii) (X_t) has cadlag trajectories.

Examples:

- combination of Brownian motion with drift and compound Poisson process
- processes with infinite number of jumps

Simulation

Although Lévy processes allow to build more realistic models we need to pay a price for increased complexity of computation. In application we can rarely use analytical methods for option pricing so numerical methods are unavoidable.
In order to apply Lévy processes one need to have efficient simulation methods.

Monte Carlo for stochastic processes

For stochastic processes one needs to simulate many trajectories of the process and obtain estimates of densities or quantiles.
Each trajectory is approximated on the discrete number of points.

Computer representation of stochastic process

Set a grid of $I+1$ time points on the interval $[t_0, T]$:

$$t_0 < t_1 < ... < t_I = T$$

where $t_i = t_0 + i\tau$ for $i = 0, 1, ..., I$ and $\tau = (T - t_0)/I$.
For each point t_i set value of the process X_{t_i}. The set of values $X_{t_0}, X_{t_1}, ..., X_{t_I}$ one trajectory of the process.
Repeat this procedure M times to obtain M trajectories of the process.

$$
\begin{array}{cccccc}
X_{t_0}^1 & X_{t_1}^1 & X_{t_2}^1 & \cdots & X_{t_{I-1}}^1 & X_{t_I}^1 \\
X_{t_0}^2 & X_{t_1}^2 & X_{t_2}^2 & \cdots & X_{t_{I-1}}^2 & X_{t_I}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
X_{t_0}^M & X_{t_1}^M & X_{t_2}^M & \cdots & X_{t_{I-1}}^M & X_{t_I}^M
\end{array}
$$

Each row represents approximation of one trajectory.
Each column represents approximation of distribution of the process in particular time point.

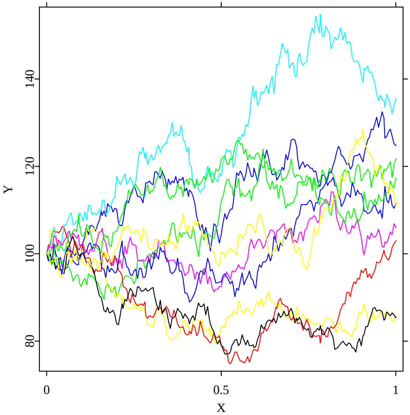


Figure 14: 10 paths of simulation of asset prices in Black-Scholes model BStrajectories.xpl

In order to calculate option price with Monte Carlo method generate sufficiently many trajectories of the possible asset's prices. Set the option price as a discounted value of the mean of the payoff.

$$C^{MC}(K) = e^{-rT} \frac{1}{M} \sum_{i=1}^{M} \max(S^i_{t_l} - K, 0)$$

where: $S^i_{t_l}$ is a simulated price of the asset in time point $t_l = T$

Simulation of Wiener process

- divide time interval $[0, T]$ in $I+1$ fixed time points $0 = t_0 < t_1 < \ldots < t_I = T$
- set $W_0 = 0$
- simulate I standard normal variables $N_1, \ldots, N_I$
- set $\Delta W_i = N_i \sqrt{t_i - t_{i-1}}$
- set $W_{t_i} = \sum_{k=1}^{i} \Delta W_k$
- repeat whole procedure M times

Simulation of Poisson process

Since the process $(N_t, t \geq 0)$ is defined by

$$N_t = \sum_{n \geq 1} \mathbf{1}_{t \geq T_n}$$

the algorithm for simulation is following:

- divide time interval $[0, T]$ in $I + 1$ fixed time points $0 = t_0 < t_1 < ... < t_I = T$ and set $N_0 = 0$
- simulate T_k from $\exp(\lambda)$ while $\sum_{i=1}^{k} T_i < T$
- set $N_{t_i} = \sup\{k : \sum_{j=1}^{k} T_j < t_i\}$
- repeat whole procedure M times

Simulation of Poisson process

Improved algorithm

- divide time interval $[0, T]$ in $I+1$ fixed time points $0 = t_0 < t_1 < ... < t_I = T$ and set $N_0 = 0$
- simulate T_k from $\mathrm{Poiss}(\lambda T)$ the number of jumps N
- simulate N uniformly distributed variables on the interval $[0, T]$ (They correspond to to the jumps time)
- set $N_{t_i} = \sup\{k : \sum_{j=1}^{k} U_j < t_i\}$
- repeat whole procedure M times

The improved algorithm for simulating Poisson process is based on two properties

- the number of jumps on the interval $[0, T]$ has Poisson distribution with parameter λT
- Conditionally on N_T the exact moments of jumps on the interval $[0, T]$ have the same distribution as N_T independent random numbers uniformly distributed on this interval. They need to rearranged in increasing order.

Simulation of compound Poisson process

- divide time interval $[0, T]$ in $I + 1$ fixed time points $0 = t_0 < t_1 < ... < t_I = T$ and set $C_0 = 0$
- generate total number of jumps N and jump times $J_1, J_2, ..., J_N$ like in Poisson process case
- simulate N random variables $Y_1, Y_2, ..., Y_N$ from the given distribution $\frac{\nu}{\lambda}$
- set $C_{t_i} = \sum_{j=0}^{N_{t_i}} Y_j$ where $Y_0 = 0$
- repeat whole procedure M times

Simulation of simple Lévy process

A simple Lévy process with characteristic triplet (a, ν, γ)

$$X_t = \gamma t + aW_t + C_t$$

can be approximated with following algorithm

- divide time interval $[0, T]$ in $I + 1$ fixed time points $0 = t_0 < t_1 < ... < t_I = T$ and set $X_0 = 0$
- generate Wiener process W_t and compound Poisson process C_t
- set $X_{t_i} = aW_{t_i} + C_{t_i} + \gamma(t_i - t_{i-1})$
- repeat whole procedure M times

Monte Carlo for Merton model

In Merton model:

$$S = S_0 \exp\{X_t\} = S_0 \exp\{\gamma t + \sigma W_t + \sum_{i \geq 1}^{N_t} Y_i\}$$

one has simple Lévy process as a sum of Wiener process with drift and compound Poisson process.
Using techniques for simulation of simple Lévy processes it is easy to obtain simulated path of asset's prices in Merton model by simple exponential transformation.

Building and simulating other Lévy process

Not every Lévy process can be obtained as simple sum of compound Poisson process and Wiener process with drift. There is a huge class of Lévy processes that have infinitely many jumps. Most of them are not easily tractable and therefore they can hardly be applied. However there are some particular cases where this kind of processes can be taken into consideration.

Building Lévy Processes

There are three convenient ways to define Lévy processes in parametric way.

- subordinating Brownian motion with independent Lévy process
- directly specifying measure
- specify the density of increments in a given time scale as infinitely divisible density

Subordination

The Lévy process S_t with monotonic increasing paths is called subordinator.
Let $(0, \rho, b)$ be a generating triplet for S_t. Then for each $u \leq 0$ moment generating function of S_t has a form:
$E(e^{uS_t}) = e^{tl(u)}$
where:
$l(u) = bu + \int_0^\infty (e^{ux} - 1)\rho(dx)$ is called Laplace exponent

Subordination

Let X_t be a Lévy processes with triplet (a, ν, γ) and characteristic exponent $\Psi(u)$ and S_t is subordinator with Laplace exponent $l(u)$ and triplet $(0, \rho, b)$.

The process $Y_t \stackrel{\mathrm{def}}{=} X_{S_t}$ is Lévy processes.
It's characteristic function is given by:
$\mathsf{E}(e^{\mathbf{i}uY_t}) = e^{tl(\Psi(u))}$

Subordination

It is also possible to find the triplet (a^Y, ν^Y, γ^Y) of Y_t.
$a^Y = ba$
$\nu^Y(B) = b\nu(B) + \int_0^\infty p_s^X(B)\rho(ds)$
$\gamma^Y = b\gamma + \int_0^\infty \rho(ds) \int_{|x|\leq 1} p_s^X(dx)$
where
p_t^X is the probability distribution of X_t

Since we need to specify p_t^X the Brownian motion is a natural candidate for X_t.
We will construct new Lévy processes by subordination of Wiener process with drift μ and volatility σ
$L_t = \sigma W_{S_t} + \mu S_t$

Generating the subordinated Brownian motion

Since many processes are based on Brownian subordination it is important to know how to simulate them.
Algorithm for simulating subordinated Brownian motion.

- divide time interval $[0, T]$ in $I+1$ fixed time points $0 = t_0 < t_1 < ... < t_I = T$ and set $X_0 = 0$
- simulate the increments of subordinator $\Delta S_i = S_{t_i} - S_{t_{i-1}}$
- simulate I standard normal variables $N_1, ..., N_I$
- set $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + \mu \Delta S_i$, where σ is volatility μ is a drift
- set $X_{t_i} = \sum_{k=1}^{i} \Delta X_k$

Example

Consider the Lévy measure of the form:

$$\rho(x) = \frac{ce^{-\lambda x}}{x}\mathbf{1}_{x>0}$$

where c and λ are positive.

Probability density of such a process is given as:

$$p_t(x) = \frac{\lambda^{ct}}{\Gamma(ct)}x^{ct-1}e^{-\lambda x}\mathbf{1}_{x>0}$$

This process is called **gamma** process and is a subordinator.

Brownian subordination of gamma process is called **variance gamma** process.

Example

Consider the Lévy measure of the form:

$$\rho(x) = \frac{ce^{-\lambda x}}{x^{3/2}} \mathbf{1}_{x>0}$$

where c and λ are positive.

Probability density of such a process is given as:

$$p_t(x) = \frac{ct}{x^{3/2}} e^{-\lambda x - \pi c^2 t^2 / x + 2ct\sqrt{\pi\lambda}} \mathbf{1}_{x>0}$$

This process is called **inverse gaussian** process and is a subordinator.

Brownian subordination of inverse Gaussian process is called **normal inverse Gaussian** process.

Stable process

Stable distribution is the distribution with characteristic function:

$$\log\phi(t)=\begin{cases}-\sigma^\alpha|t|^\alpha\{1-i\beta\,\mathtt{sign}(t)\tan\frac{\pi\alpha}{2}\}+i\mu t, & \alpha\neq 1,\\[2ex] -\sigma|t|\{1+i\beta\,\mathtt{sign}(t)\frac{2}{\pi}\log|t|\}+i\mu t, & \alpha=1.\end{cases}$$

Stable processes are process with stable distribution. For $\alpha=2$ stable process is a Brownian motion.

Generalized hyperbolic process

Probability density function of generalized hyperbolic distribution has a form:

$$p(x) = C(\delta^2 + (x-\mu)^2)^{\frac{\lambda}{2}-\frac{1}{4}} K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})e^{\beta(x-\mu)}$$

where: $C = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})}$

Lévy process (X_t) is called Generalized Hyperbolic Lévy Motion, when X_1 has generalized hyperbolic distribution.

Remark

Let (X_t) be a Generalized Hyperbolic Lévy Motion. Then X_t doesn't need to have for $t \neq 1$ generalized hyperbolic distribution.

Calibration Problem

Risks in option pricing I

Model risk:

Different models can have the same plain vanilla prices but significantly varying exotics prices.

(see e.g. [Schoutens et al. (2004)] or [Cont (2005)])

Risks in option pricing II

Calibration risk:

A model calibrated to plain vanillas with respect to different error functionals can lead to significantly varying exotics prices although the plain vanilla prices are similar.

How should the error be measured?

Questions about calibration risk

- How big is calibration risk?
- Which factors influence calibration risk?
  - option type
  - time to maturity
  - goodness of fit of the model
- Model risk $< - >$ calibration risk

Overview

- motivation✓
- model and data
- calibration
- exotic options
  - up and out calls
  - down and out puts
  - cliquets
  - model risk
- conclusion

The Heston model

The price process is given by

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^{(1)}$$

where the volatility process is modelled by a square-root process:

$$dV_t = \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^{(2)},$$

and W^1 and W^2 are Wiener processes with correlation ρ.

The Heston model II

The volatility process (V_t) remains positive if

$$\xi\eta > \frac{\theta^2}{2}$$

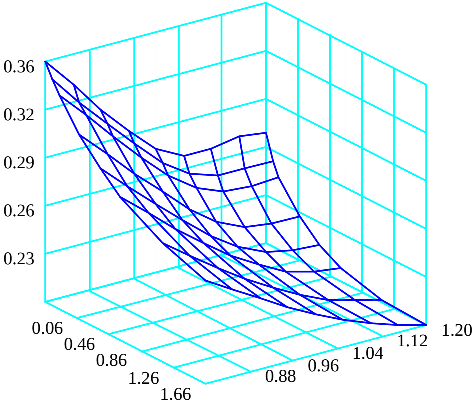


Figure 15: Implied volatility surface of the Heston model for $\xi = 1.0$, $\eta = 0.15$, $\rho = -0.5$, $\theta = 0.5$ and $v_0 = 0.1$.
(Left axis: time to maturity, right axis: moneyness)

The Bates model

In this model, the price process is given by

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t} dW_t^{(1)} + dZ_t \\ dV_t &= \xi(\eta - V_t)dt + \theta\sqrt{V_t} dW_t^{(2)}\end{aligned}$$

where Z is a compound Poisson process.

This model extends the Heston model and has three more parameters.

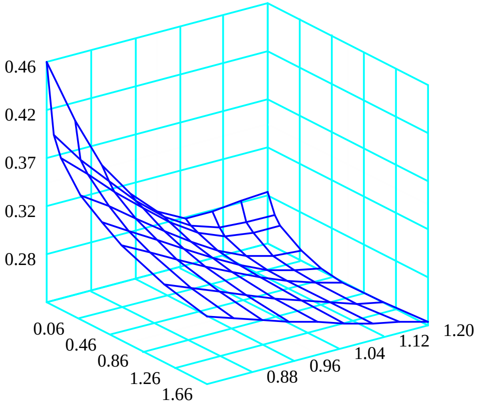


Figure 16: Implied volatility surface of the Bates model for $\lambda = 0.5$, $\delta = 0.2$, $\overline{k} = -0.1$, $\xi = 1.0$, $\eta = 0.15$, $\rho = -0.5$, $\theta = 0.5$ and $v_0 = 0.1$. (Left axis: time to maturity, right axis: moneyness)

Data

- Eurex settlement volatilities of European options
- underlying : dax
- time period: April 2003 - March 2004
- risk free interest rate: Euribor
- no explicit dividends because dax is performance index

Data II

Arbitrage:
The implied volatility surfaces have been preprocessed by a method of [Fengler (2005)] in order to eliminate arbitrage.

Illiquidity:
Only options with moneyness $m \in [0.75, 1.35]$ for small times to maturity $T \leq 1$ have been considered because of illiquidity.

	mean number of maturities	mean number of obervations	mean money-ness range
short maturities $(0.25 \leq T < 1.0)$	3.06	64	0.553
long maturities $(1.0 \leq T)$	5.98	76	0.699
total	9.04	140	0.649

Table 4: Description of the implied volatility surfaces.

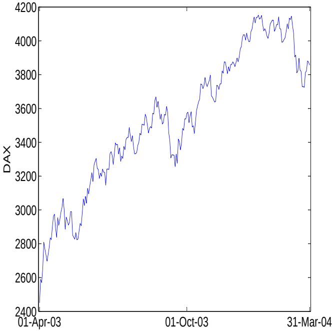


Figure 17: DAX and ATM implied volatility with 1 year to maturity on the trading days from 01 April 2003 to 31 March 2004.

Error functionals I

For the minimization we consider the following four objective functions based on the root weighted square error:

$$\text{absolute prices: } AP \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i (P_i^{mod} - P_i^{mar})^2}$$

$$\text{relative prices: } RP \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i \left(\frac{P_i^{mod} - P_i^{mar}}{P_i^{mar}}\right)^2}$$

Error functionals II

$$\text{absolute iv: } AI \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i (IV_i^{mod} - IV_i^{mar})^2}$$

$$\text{relative iv: } RI \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i (\frac{IV_i^{mod} - IV_i^{mar}}{IV_i^{mar}})^2}$$

where *mod* refers to a model quantity and *mar* to a quantity observed on the market, P to an OTM price and IV to an implied volatility.

Weights

$$w_i \stackrel{\text{def}}{=} \frac{1}{n_{mat} n_{str}^i}$$

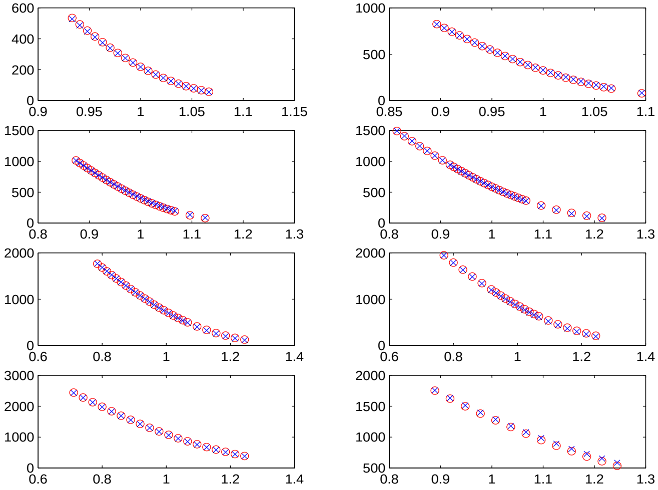
where n_{mat} denotes the number of maturities and n_{str}^i denotes the number of strikes with the same maturity as observation i.

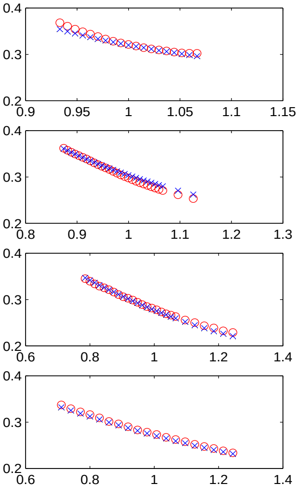
(calibration design)

Calibration method

The error functionals are minimized with respect to the model parameters by a global stochastic minimization routine.

The plain vanilla prices are calculated by a method of Carr and Madan that is based on the FFT.


600
400
200
0
0.9 0.95 1 1.05 1.1 1.15
1000
500
0
0.85 0.9 0.95 1 1.05 1.1
1500
1000
500
0
0.8 0.9 1 1.1 1.2 1.3
1500
1000
500
0
0.8 0.9 1 1.1 1.2 1.3
2000
1000
0
0.6 0.8 1 1.2 1.4
2000
1000
0
0.6 0.8 1 1.2 1.4
3000
2000
1000
0
0.6 0.8 1 1.2 1.4
2000
1500
1000
500
0.8 0.9 1 1.1 1.2 1.3


0.4
0.3
0.2
0.9
0.95
1
1.05
1.1
1.15
0.85
1.3
0.8
0.6
0.8
1.2
1.4

objective fct. \ mean	AP	RP $[E^{-2}]$	AI $[E^{-2}]$	RI $[E^{-2}]$
AP	7.3	9.7	0.81	3.1
RP	11.	6.1	0.74	2.9
AI	9.4	7.3	0.68	2.6
RI	8.8	7.0	0.70	2.5

Table 5: Calibration errors in the Heston model for 51 days.

	ξ	η	θ	ρ	V_0
AP	0.87	0.07	0.34	-0.82	0.07
	(0.48)	(0.02)	(0.08)	(0.08)	(0.02)
RP	1.38	0.07	0.44	-0.74	0.08
	(0.35)	(0.02)	(0.06)	(0.03)	(0.02)
AI	1.32	0.07	0.43	-0.77	0.08
	(0.40)	(0.02)	(0.06)	(0.04)	(0.02)
RI	1.20	0.07	0.41	-0.75	0.08
	(0.35)	(0.02)	(0.06)	(0.05)	(0.02)

Table 6: Mean parameters (std.) in the Heston model for 51 days.

mean objective fct.	AP	RP $[E^{-2}]$	AI $[E^{-2}]$	RI $[E^{-2}]$
AP	7.0	13.	0.76	2.8
RP	12.	5.1	0.67	2.6
AI	8.9	6.4	0.60	2.3
RI	8.7	6.2	0.62	2.2

Table 7: Calibration errors in the Bates model for 51 days.

	ξ	η	θ	ρ	V_0	λ	$\overline{k}$	δ
AP	0.92	0.07	0.33	-0.94	0.07	0.33	0.07	0.08
	(0.50)	(0.02)	(0.08)	(0.07)	(0.02)	(0.21)	(0.03)	(0.06
RP	1.56	0.07	0.45	-0.89	0.08	0.54	0.05	0.08
	(0.47)	(0.02)	(0.07)	(0.07)	(0.02)	(0.23)	(0.03)	(0.06
AI	1.43	0.07	0.43	-0.95	0.07	0.50	0.06	0.09
	(0.44)	(0.02)	(0.06)	(0.06)	(0.02)	(0.22)	(0.03)	(0.04
RI	1.36	0.07	0.41	-0.93	0.07	0.52	0.05	0.08
	(0.44)	(0.02)	(0.07)	(0.09)	(0.02)	(0.26)	(0.04)	(0.08

Table 8: Mean parameters (std.) in the Bates model for 51 days.

Monte Carlo simulation

We generate 1000000 paths by Euler discretization.

We approximate the continuous maximum by a discrete maximum for 250 time steps a year.

	$T = 1$ $[E^{-2}]$	$T = 2$ $[E^{-2}]$	$T = 3$ $[E^{-2}]$
up and out calls	0.17	0.10	0.08
down and out puts	0.18	0.11	0.08
cliquet options	0.06	0.05	0.05

Table 9: Maximal relative standard error in MC simulations in Heston model.

Barrier options

The prices of up and out calls are given by

$$\exp(-rT)\,\mathsf{E}[(S_T - K)^+ \mathbf{1}_{\{M_T < B\}}]$$

where

$$M_T \stackrel{\mathrm{def}}{=} \max_{0 \le t \le T} S_t .$$

Barrier options II

We consider for the barrier B and the strike K

$$B = 1 + T * 0.2$$
$$K = 1 - T * 0.1$$

where $T = 1, 2, 3$ denotes time to maturity.

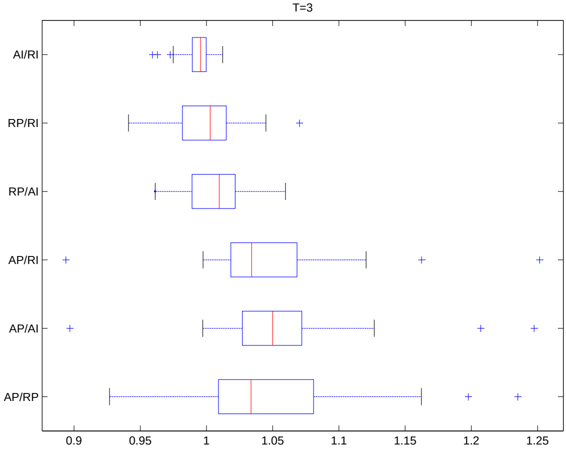


Figure 18: Prices of the up and out calls in the Heston model.

up and out calls in Heston

- two groups: AP and RP,AI,RI
- difference between groups bigger for higher time to maturity
- AP roughly 5% more expensive than the rest ($T = 3$)
- high variance between AP and RP
- small variance between AI and RI

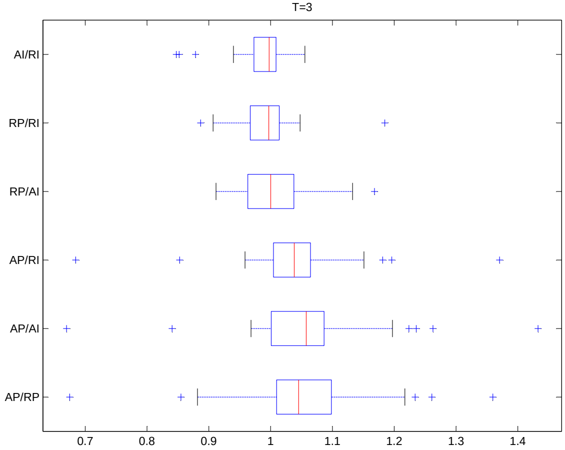


Figure 19: Prices of the up and out calls in the Bates model.

up and out calls in Bates

Qualitative results similar to the results in the Heston model.
But quantitatively more significant, i.e. the differences are bigger.

possible explanations:

- calibration problems (3 more parameters)
- the higher prices in the Bates model imply higher calibation risk. (connection to model risk)

Barrier options III

We also consider down and out puts with barrier B and strike K

$$B = 1 - T * 0.2$$
$$K = 1 + T * 0.1$$

where $T = 1, 2, 3$ denotes time to maturity.

down and out puts in Heston

- two groups: AP and RP,AI,RI
- difference between groups bigger for higher time to maturity
- AP roughly 4% cheaper than the rest ($T = 3$)
- high variance between AP and the rest

Results similar to up and out call but different sign/direction.

up and out calls in Bates

Qualitative results similar to the results in the Heston model.
But here calibration risk is not bigger only the variance is higher.

Cliquet options

The prices of cliquets are given by

$$\exp(-rT)\,\mathsf{E}[H]$$

where the payoff H is given by

$$H \stackrel{\text{def}}{=} \min(c_g, \max(f_g, \sum_{i=1}^{N} \min(c_l^i, \max(f_l^i, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})))).$$

Here c_g (f_g) is a global cap (floor) and c_g^i (f_g^i) is a local cap (floor) for the period $[t_{i-1}, t_i]$.

Cliquet options II

We consider three periods with $t_i = \frac{i}{3}T$ $(i = 0, \ldots, 3)$ and the caps and floors are given by

$$
\begin{aligned}
c_g &= \infty \\
f_g &= 0 \\
c_l^i &= 0.08, \;\; i = 1, 2, 3 \\
f_l^i &= -0.08, \;\; i = 1, 2, 3
\end{aligned}
$$

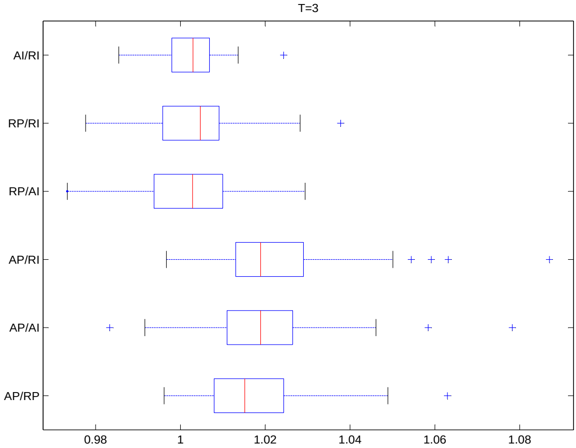


Figure 20: Prices of the cliquets in the Heston model.

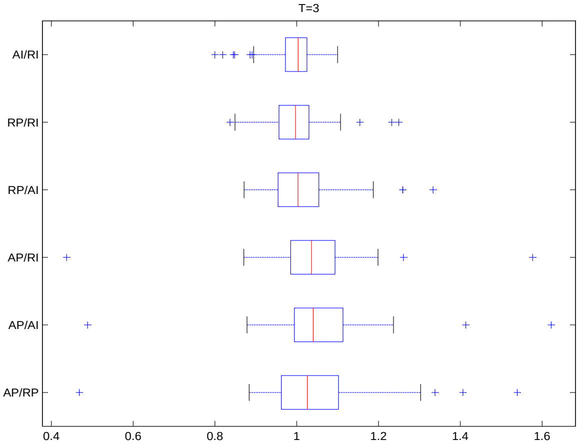


Figure 21: Prices of the cliquets in the Bates model.

cliquets

- two groups: AP and RP,AI,RI
- difference between these groups than for barriers
- Heston: AP roughly 2% more expensive than the rest ($T = 3$)
- Bates: similar to Heston only higher variance
- small variance (compared to barriers)

model risk

The prices in the Bates model are higher than the corresponding prices in the Heston model for barrier options and lower for cliquets (for ALL times to maturity and ALL error functionals).

- up and out calls: $4 - 14\%$
- down and out puts: $10 - 20\%$
- cliquets: $14 - 25\%$

Results suggest higher model risk than the results of [Schoutens et al. (2004)].

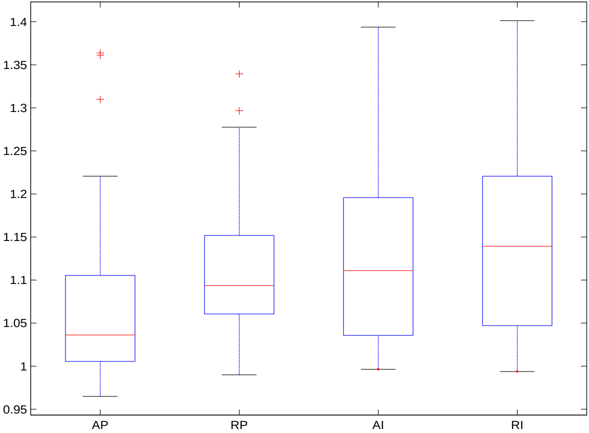


Figure 22: Bates prices over Heston prices for up and out calls.

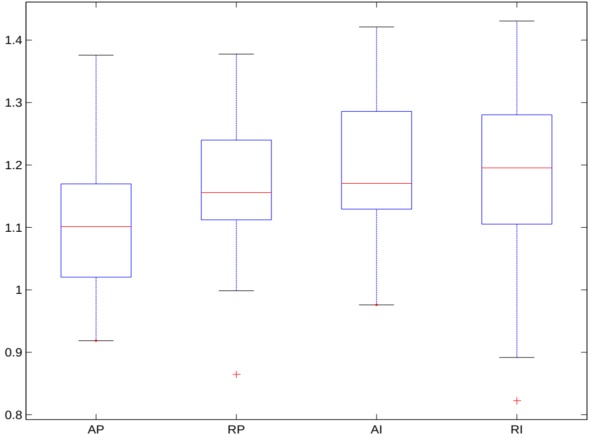


Figure 23: Bates prices over Heston prices for down and out puts.

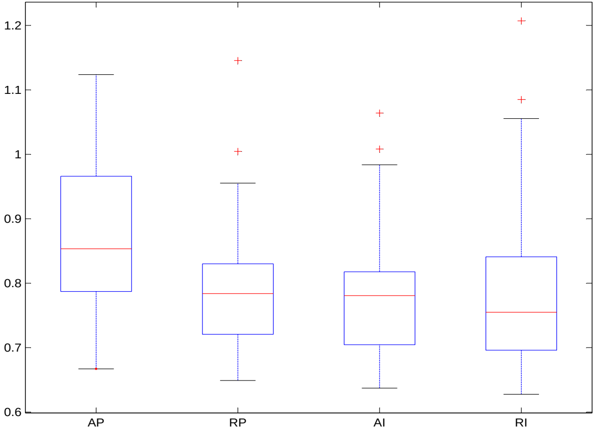


Figure 24: Bates prices over Heston prices for cliquets.

model risk II

Model risk and calibration risk are not independent.
Model risk depends on the error functional minimized.

It is smallest for AP error functional. It is highest for RI error functional.

Conclusion

- there is calibration risk
- two groups: AP and RP,AI,RI
- calibration risk grows with time to maturity
- calibration risk differs for different option types (barriers $>$ cliquets)
- calibration risk is bigger for Bates than for Heston
- model and calibration are dependent
- for each product a model and an error functional should be chosen

Dynamics of the Implied Volatility Surface

Aims

Model and estimate implied volatility surfaces (IVS) for

- trading
- hedging of derivative positions
- risk management.

In these contexts the IVS acts as a very **high-dimensional state variable**.
Practice requires a **low-dimensional representation** of the IVS.

Challenges

- Large number of observations ($>$ 2 million contracts, $>$ 5 000 observations per day).
- Data appear in ‘strings’.
- Strings are not locally fixed, but ‘move’ through the observation space (expiry effect).
- In the moneyness dimension observations may be missing in certain sub-regions for some dates i.

Degenerated Design

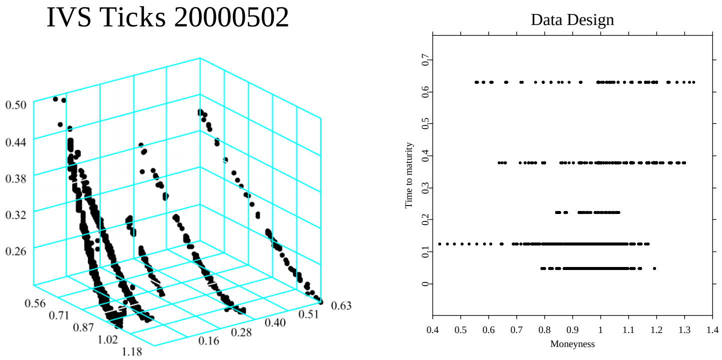


Figure 25: Left panel: call and put implied volatilities observed on 20000502. Right panel: data design on 20000502; ODAX, difference-dividend correction according to Hafner and Wallmeier (2001) applied.

Purpose

A **modelling strategy** in terms of a **dynamic semiparametric factor model (DSFM)** for the (log)-IVS
$Y_{i,j}$ ($i = $ day, $j = $ intraday):

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) \quad . \tag{9}$$

Here $m_l(X_{i,j})$ are smooth factor functions and $\beta_{i,l}$ is a multivariate (loading) time-series.

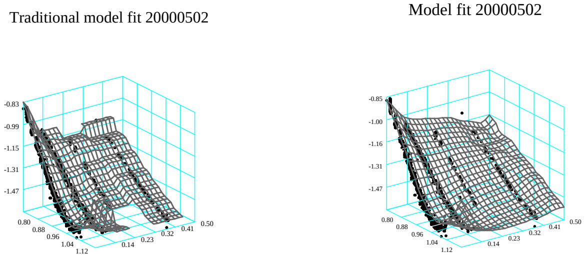


Figure 26: *Traditional model (Nadaraya-Watson estimator) and semi-parametric factor model fit for 20000502. Bandwidths for both estimates* $h_1 = 0.03$ *for the moneyness and* $h_2 = 0.08$ *for the time to maturity dimension.*

Overview

1. aims and generic challenges✓
2. implied volatilities
3. short literature review
4. model
5. algorithm
6. results
7. application
8. outlook

Implied volatilities

Black and Scholes (1973) (BS) formula prices European options under the assumption that the asset price S_t follows a geometric Brownian motion with constant drift and constant volatility coefficient σ:

$$C_t^{BS} = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2) \ ,$$

where $d_{1,2} = \frac{\ln(S_t/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$. $\Phi(u)$ is the CDF of the standard normal distribution, r a constant interest rate, $\tau = T - t$ time to maturity, K the strike price.

Implied volatilities

Volatility $\hat{\sigma}$ as *implied* by observed market prices $\tilde{C}_t$:

$$\hat{\sigma}: \quad \tilde{C}_t - C_t^{BS}(S_t, K, \tau, r, \hat{\sigma}) = 0 \, .$$

Unlike assumed in the BS model, $\hat{\sigma}_t(K, \tau)$ exhibits **distinct, time-dependent** functional patterns across K (**smile or smirk**), and a **term-structure** $T - t$: Thus $\hat{\sigma}_t(K, \tau)$ is interpreted as a **random surface**: the implied volatility surface (IVS).

Related work

One strand of literature models IVS 'slices' using PCA:

- Alexander (2001) analyzes fixed strike deviations,
- Skiadopoulos et al. (1999) explore the smile in different maturity buckets,
- Avellaneda and Zhu (1997); Fengler et al. (2002) investigate the term structure.

Related work

Recently, a more comprehensive surface perspective is adopted:

- Fengler et al. (2003) propose a simultaneous decomposition of maturity groups in a **common principal components** framework.
- Cont and da Fonseca (2002) employ the **Karhunen und Loève** decomposition.

This literature does not properly cope with the degenerated design. Estimates are necessarily **biased**.

The semiparametric factor model

Consider DSFM for the IVS:

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) \quad , \tag{10}$$

$Y_{i,j}$ is log IV, i denotes the trading day ($i = 1, \ldots, I$),
$j = 1, \ldots, J_i$ is an index of the traded options on day i.
$m_l(\cdot)$ for $l = 0, \ldots, L$ are basis functions in covariables $X_{i,j}$,
and β_i are time dependent factors.

For $m_l(\cdot)$, $l = 0, \ldots, L$ consider two different set-ups in $X_{i,j}$:

(A) $X_{i,j}$ is a two-dimensional vector containing time to maturity $\tau_{i,j}$ and forward moneyness, $\kappa_{i,j} = \frac{K}{F(t_{i,j})}$, i.e. strike K divided by futures price $F(t_{i,j}) = S_{t_{i,j}} \exp(r_{\tau_{i,j}} \tau_{i,j})$

(B) as in (A) but with one-dimensional $X_{i,j}$ that only contains $\kappa_{i,j}$.

Here, we focus on (A).

Space and time smoothing

Define estimates of $\widehat{m}_l$ and $\widehat{\beta}_{i,l}$ with $\widehat{\beta}_{i,0} \stackrel{\text{def}}{=} 1$, as minimizers of:

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i} \int \left\{ Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(u) \right\}^2 K_h(u - X_{i,j}) \, du, \qquad (11)$$

where K_h denotes a two dimensional product kernel, $K_h(u) = k_{h_1}(u_1) \times k_{h_2}(u_2), \quad h = (h_1, h_2)$ with a one-dimensional kernel $k_h(v) = h^{-1} k(h^{-1} v)$.

Replace in (21) $\widehat{m}_l$ by $\widehat{m}_l + \delta g$ and $\widehat{\beta}_{i,l}$ by $\widehat{\beta}_{i,l} + \delta$. Take derivatives wrt δ, $(1 \le l' \le L, 1 \le i \le I)$:

$$\sum_{i=1}^{I} J_i \widehat{\beta}_{i,l'} \widehat{q}_i(u) = \sum_{i=1}^{I} J_i \sum_{l=0}^{L} \widehat{\beta}_{i,l'} \widehat{\beta}_{i,l} \widehat{p}_i(u) \widehat{m}_l(u), \qquad (12)$$

$$\int \widehat{q}_i(u) \widehat{m}_{l'}(u)\, du = \sum_{l=0}^{L} \widehat{\beta}_{i,l} \int \widehat{p}_i(u) \widehat{m}_{l'}(u) \widehat{m}_l(u)\, du, \qquad (13)$$

$$\widehat{p}_i(u) = \frac{1}{J_i} \sum_{j=1}^{J_i} K_h(u - X_{i,j}),$$

$$\widehat{q}_i(u) = \frac{1}{J_i} \sum_{j=1}^{J_i} K_h(u - X_{i,j}) Y_{i,j}.$$

Model characteristics

Consider the case $L = 0$: the log-implied volatilities $Y_{i,j}$ are approximated by a surface $\widehat{m}_0$ not depending on day i. Then,

$$\widehat{m}_0(u) = \frac{\sum_{i,j} K_h(u - X_{i,j}) Y_{i,j}}{\sum_{i,j} K_h(u - X_{i,j})},$$

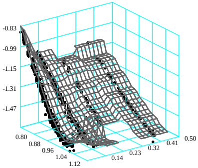
$\widehat{m}_0$ is equal to the Nadaraya-Watson estimate based on the pooled sample of all days.

Model characteristics

Consider a fixed day i and $L = 0$:,

$$\widehat{m}_0^{(i)}(u) = \frac{\sum_{j=1}^{J_i} K_h(u - X_{i,j}) Y_{i,j}}{\sum_{j=1}^{J_i} K_h(u - X_{i,j})},$$

Traditional model fit 20000502



Model characteristics

IVS's are fitted in *neighborhoods* of the observed design points $X_{i,j}$, i.e.

- we do not fit the surface on the whole design space on each day (as in a functional PCA (fPCA), Ramsay and Silverman (1997)).
- we circumvent global fits and thus avoid large bias effects caused by the degenerated string design.

Model characteristics

In fPCA factors are eigenfunctions of a covariance operator. Here, the norm:

$$\int f^2(u)\hat{p}_i(u)du \; ,$$

changes each day i, where $\hat{p}_i(u) = J_i^{-1}\sum_{j=1}^{J_i} K_h(u - X_{i,j})$.
Eigenfunctions $\widehat{m}_l$ may not be nested for increasing L:
Hence, the $\widehat{m}_l$ cannot be calculated iteratively, i.e. by moving from $L-1$ components to L components, and so forth.

Model characteristics

In the DSFM framework the IVS's are approximated by surfaces moving in the function space

$$\{\widehat{m}_0 + \sum_{l=1}^{L} \alpha_l \widehat{m}_l : \ \alpha_1, \ldots, \alpha_L \in \mathbb{R}\}.$$

The estimates $\widehat{m}_l$ are not uniquely defined: they can be replaced by estimates that span the same affine space.
Natural choice: orthogonalize $\widehat{m}_l$ in an appropriate function space.
Order the resulting functions according to maximum variance in $\widehat{\beta}_l$.

Orthogonalization

Replace:

$$\widehat{m}_0 \quad \text{by} \quad \widehat{m}_0^{new} = \widehat{m}_0 - \gamma^\top \Gamma^{-1} \widehat{m}$$

$$\widehat{m} \quad \text{by} \quad \widehat{m}^{new} = \Gamma^{-1/2} \widehat{m}$$

$$\widehat{\beta}_i \quad \text{by} \quad \widehat{\beta}_i^{new} = \Gamma^{1/2}(\widehat{\beta}_i + \Gamma^{-1}\gamma)$$

where:

$\widehat{m} = (\widehat{m}_1, ..., \widehat{m}_L)^\top$, $\widehat{\beta}_i = (\widehat{\beta}_{i,1}, ..., \widehat{\beta}_{i,L})^\top$, $\widehat{p}(u) = \frac{1}{I}\sum_{i=1}^{I} \widehat{p}_i(u)$

Γ is (LxL) matrix with $\Gamma_{l,l'} = \int \widehat{m}_l(u)\widehat{m}_{l'}(u)\widehat{p}(u)du$

γ is (Lx1) vector with $\gamma_l = \int \widehat{m}_0(u)\widehat{m}_l(u)\widehat{p}(u)du$.

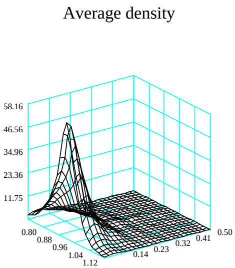


Figure 27: *The average density* $\widehat{p}(u)$

Ordering

Define matrix B with $B_{l,l'} = \sum_{i=1}^{I} \widehat{\beta}_{i,l}\widehat{\beta}_{i,l'}$ and $Z = (z_1, ..., z_L)$ where z_1,...,z_L eigenvectors of B.

Replace:

$$\widehat{m} \quad \text{by} \quad \widehat{m}^{new} = Z^\top \widehat{m}$$

$$\widehat{\beta}_i \quad \text{by} \quad \widehat{\beta}_i^{new} = Z^\top \widehat{\beta}_i$$

The orthonormal basis $\widehat{m}_1, ..., \widehat{m}_L$ is chosen such that $\sum_{i=1}^{I} \widehat{\beta}_{i,1}^2$ is maximal and given $\widehat{\beta}_{i,1}, \widehat{m}_0, \widehat{m}_1$ the quantitiy $\sum_{i=1}^{I} \widehat{\beta}_{i,2}^2$ is maximal and so forth.

Algorithm

The algorithm exploits equations (12) and (13) iteratively:

1. for an appropriate initialization of
   $\beta_{l,i}^{(0)}, \quad i = 1, \ldots, I, \quad l = 1, \ldots, L$
   get an initial estimate of $\widehat{m}^{(0)} = (\widehat{m}_0, ..., \widehat{m}_L)^\top$
2. update $\beta_i^{(1)}, \quad i = 1, \ldots, I,$
3. estimate $\widehat{m}^{(1)}$.
4. go to step 2.

until minor changes occur during the cycle.
Optimization implemented in XploRe, DSFM.xpl, Härdle et al. (2000).

Data Overview

	Min.	Max.	Mean	Median	Stdd.	Skewn.	Kurt.
T. to mat.	0.028	2.002	0.142	0.086	0.166	3.658	21.449
Moneyness.	0.287	3.367	0.996	0.997	0.114	0.686	12.026
IV	0.040	0.799	0.297	0.265	0.105	1.289	4.489

Table 10: *Summary statistics from 199901 to 200302. Source: EU-REX, ODAX, stored in the SFB 649 FEDC.*

$J_i \approx 5\,200$ observations per day
total time series has $I \approx 1000$ days.
$N = IJ_i \approx 2.8$ million contracts,

Model selection

For a data-driven choice of bandwidths we propose a weighted AIC since the distribution of observations is very unequal:

$$\Xi_{AIC_1} = \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j})\}^2 w(X_{i,j}) \exp\{\frac{2L}{N} K_h(0) \int w(u)du\},$$

alternatively (computationally easier):

$$\Xi_{AIC_2} = \frac{1}{N} \sum_{i,j} \{Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j})\}^2 \exp\{\frac{2L}{N} K_h(0) \frac{\int w(u)du}{\int w(u)p(u)du}\}.$$

w is a given weight function. Putting $w(u) = 1$ delivers common AIC, putting $w(u) = \frac{1}{p(u)}$ give equal weight everywhere.

Model selection

For the model size (L) selection use the:

$$RV(L) = \frac{\sum_i^I \sum_j^{J_i} \{Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(X_{i,j})\}^2}{\sum_i^I \sum_j^{J_i} (Y_{i,j} - \bar{Y})^2}$$

where $\bar{Y}$ denotes the overall mean of the observations.

L	1-RV(L)	ΔRV
1	0.9638	
2	0.9739	0.0101
3	0.9822	0.0083
4	0.9830	0.0007

Table 11: *Explained variance for the model size.*

Estimation Results

We fit the model for $L = 3$, i.e. there are

- one invariant basis function $\widehat{m}_0$ and
- 3 'dynamic' basis functions $\widehat{m}_1, \widehat{m}_2, \widehat{m}_3$
- 3 time series of $\{\beta_{l,i}\}_{i=1}^{I}$ with $l = 1, 2, 3$

The bandwidths were chosen according to $AIC2$ criterion:
$h_1 = 0.03, h_2 = 0.02$

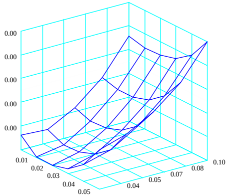


Figure 28: Ξ_{AIC_2} *dependence on the bandwidths.*

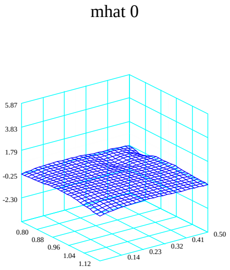


Figure 29: *Invariant basis function* $\widehat{m}_0$ *and dynamic basis function* $\widehat{m}_1$

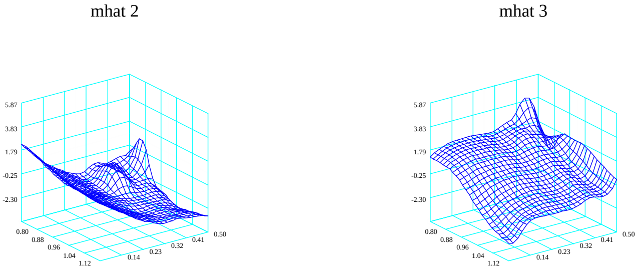


Figure 30: *Dynamic basis functions* $\widehat{m}_2$ *and* $\widehat{m}_3$

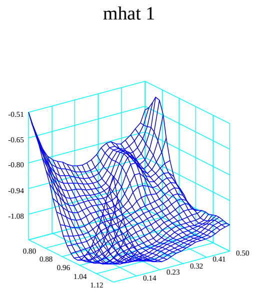


Figure 31: *The dynamic basis function* $\widehat{m}_1$

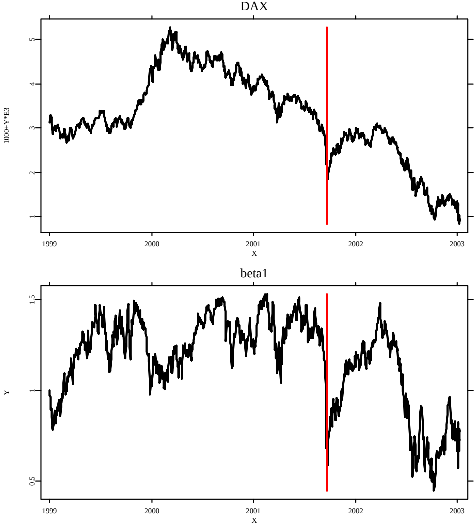


Figure 32: *DAX and time series of weights* $\widehat{\beta}_1$

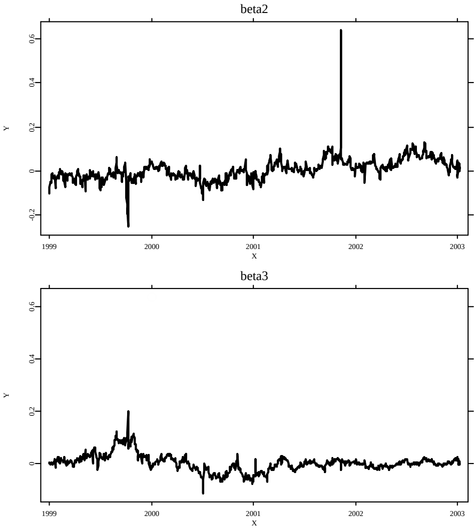


Figure 33: *Time series of weights* $\widehat{\beta}_2$ *and* $\widehat{\beta}_3$

Correlogram for $\widehat{\beta}_1$, $\widehat{\beta}_2$ and $\widehat{\beta}_3$

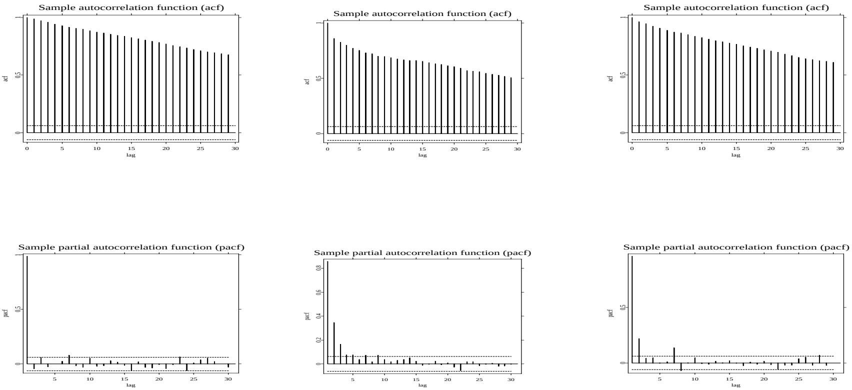


Figure 34: *acf and pacf of* $\widehat{\beta}_1$, $\widehat{\beta}_2$ *and* $\widehat{\beta}_3$ *respectively*

Testing for random walk

coeff.	lag differences	suggested break date	test-value
$\widehat{\beta}_1$	7	2001.11.09	-1.33
$\widehat{\beta}_2$	2	2001.11.09	-1.09
	3		-1.04
$\widehat{\beta}_3$	2	1999.06.08	-3.42*

Table 12: *Unitroot test in the presence of structural break. Critical values for rejecting the hypothesis of unit root are -2.88 at* 5% *significance level and -3.48 at* 1% *significance level. (*) indicate significance at* 5% *level. Lane et al. (2002)*

We model first differences of $\widehat{\beta}_1$, $\widehat{\beta}_2$ and level $\widehat{\beta}_3$ in the form

$$Y_t = (\Delta\widehat{\beta}_1, \Delta\widehat{\beta}_2, \hat{\beta}_3)^\top$$

$$Y_t = \upsilon + A_1 Y_{t-1} + A_2 Y_{t-2} +, \ldots, + A_p Y_{t-p} + \varepsilon_t$$

$Y_t = (Y_{1t}, \ldots, Y_{kt})^\top$ are vectors of the $k = 3$ endogenous variables

$\upsilon = (\upsilon_1, \ldots, \upsilon_k)^\top$ is a vector of intercept terms, A_i are $(K \times K)$ coefficient matrices

ε_t is a white noise with covariance matrix $\Sigma_\varepsilon > 0$

Order Selection Criteria

Lag	ln(FPE)	AIC	SC	HQ
0	-24.34	-15.83	-15.81	-15.82
1	-24.61	-16.10	-16.04	-16.07
2	-24.66	-16.15	-16.05*	-16.11
3	-24.68	-16.17	-16.03	-16.11*
4	-24.68	-16.17	-15.98	-16.10
5	-24.69	-16.16	-15.94	-16.08
6	-24.70*	-16.18*	-15.91	-16.08
7	-24.69	-16.18	-15.87	-16.06
8	-24.69	-16.17	-15.82	-16.04

Table 13: *VAR Lag Order Selection. * indicates lag order selected by the criterion up to a maximum order 8. We chose to apply a VAR(2) as indicated by the SC criterion.*

$$\begin{bmatrix} \Delta\widehat{\beta}_{1t} \\ \Delta\widehat{\beta}_{2t} \\ \widehat{\beta}_{3t} \end{bmatrix} = \begin{bmatrix} 0.12 & 0.22 & -0.09 \\ -0.09 & -0.57 & 0.08 \\ 0.01 & 0.03 & 0.74 \end{bmatrix} \begin{bmatrix} \Delta\widehat{\beta}_{1t,t-1} \\ \Delta\widehat{\beta}_{2t,t-1} \\ \widehat{\beta}_{3t,t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} -0.07 & 0.03 & 0.09 \\ -0.01 & -0.24 & -0.07 \\ -0.01 & 0.01 & 0.23 \end{bmatrix} \begin{bmatrix} \Delta\widehat{\beta}_{1t,t-2} \\ \Delta\widehat{\beta}_{2t,t-2} \\ \widehat{\beta}_{3t,t-2} \end{bmatrix}$$

$$+ \begin{bmatrix} \hat{u}_{1,t} \\ \hat{u}_{2,t} \\ \hat{u}_{3,t} \end{bmatrix}$$

VAR model for first difference levels, $(\Delta\widehat{\beta}_1, \Delta\widehat{\beta}_2, \hat{\beta}_3)^\top$

Model Stability

Time invariance of the model has been evaluated through the roots of the characteristic polynomial for the VAR(2) model as well as coefficient stability through the cumulative sum of squares of the residuals.

roots	modulus
0.97	0.97
$-0.27 \pm 0.4i$	0.48
$0.04 \pm 0.2i$	0.27
-0.23	0.23

Table 14: *Roots of characteristic polynomial for the VAR(2): stability condition is satisied since no root lies outside the unit circle.*

Model Stability

CUSUM-square statistic:

$$S_t = \frac{\sum_{r=k+1}^{t} W_r^2}{\sum_{r=k+1}^{T} W_r^2}$$

W_r^2 (recursive residuals) is the square one-period ahead prediction error. $r = k+1, \dots, T$ (k, the number of regressors including a constant and T, sample size.
We plot S_r together with significance level lines $E[S_r] \pm C_0$, the statistical "boundaries". C_0 depends on $T - k$ and the significance level desired, see Harvey (1990).

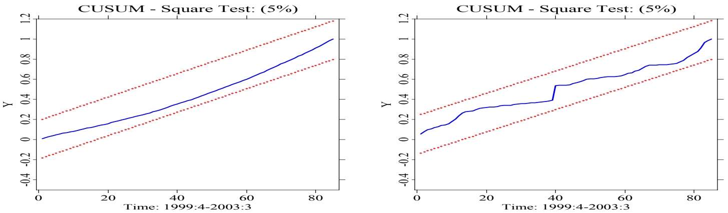


Figure 35: *CUSUM-square statistics for* Δz_1 *and* Δz_2 *equation*

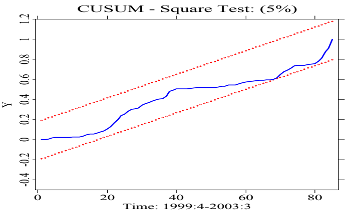


Figure 36: *CUSUM-square statistics for z_3 equation*

Coefficient stability is not rejected as all plots lies within the critical boundaries.

Hedging exotic options

Knock-out options are financial options that become worthless as soon as the underlying reaches a prespecified barrier.

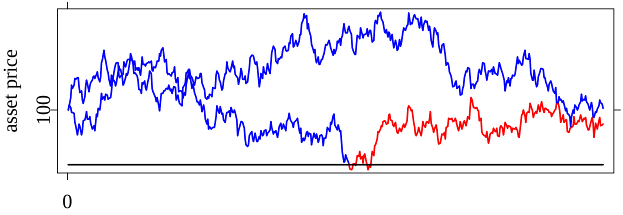


Figure 37: Example of two possible paths of asset's price. When the price hits the barrier (red) the option is no longer valid regardless further evolution of the price.

PROTECT-AKTIENANLEIHEN

DIE PROTECT-AKTIENANLEIHE Kupon p. a.	Basispreis in Euro	PROTECT-Preis in Euro	Alternativ-rückzahlung in Aktien	WKN	Verkaufskurs in %
7,5 % Deutsche Bank	60,98	49,00	82	SAL 22W	100,40
5,0 % Deutsche Telekom	15,24	12,75	328	SAL 22Y	100,20
8,0 % SAP	135,14	105,00	37	SAL 234	100,70
10,0 % TUI	16,23	12,50	308	SAL 235	*100,00*

DAS PRINZIP
Beispiel 7,5 % PROTECT-Aktienanleihe auf Deutsche Bank: Die Anleihe wird am 30. November 2005 zu 100 % zurückgezahlt, sofern die Aktie der Deutsche Bank AG im Xetra-Handelssystem bis zum 23. November 2005 nicht einmal bei bzw. unter dem PROTECT-Preis von 49,00 Euro notiert oder am 23. November 2005 über dem Basispreis von 60,98 Euro schließt. Andernfalls ist die Emittentin berechtigt, als Alternative 82 Aktien je 5.000 Euro Nominalbetrag zu liefern. Die Zinsen in Höhe von *7,5 % werden garantiert gezahlt.*

Fälligkeit:** 30. November 2005.* ***Anlagebetrag: *Nominal* 5.000 Euro oder ein Vielfaches. **Zinszahlung:** Ab 9. November 2004. **Börsenhandel:** Düsseldorf, Frankfurt, Stuttgart. Allein maßgeblich ist der Verkaufsprospekt, dem Sie auch nähere Informationen zu den Chancen und Risiken des Produktes entnehmen können. Das Anlage-Barometer stellt keine Anlageempfehlung dar und ersetzt nicht die individuelle Beratung durch Ihre Hausbank. Den Verkaufsprospekt erhalten Sie kostenlos bei der Emittentin, Sal. Oppenheim jr. & Cie. KGaA, Untermainanlage 1, 60329 Frankfurt am Main. Die Verkaufskurse werden fortlaufend an die Marktentwicklung angepaßt; Stand: 5. November 2004
Service-Telefon 069/71 34-22 33 **E-Mail** retailproducts@oppenheim.de **Internet** www.oppenheim-derivate.de ***Teletext** n-tv Tafel 819*

Figure 38: Newspaper advertisement of Sal. Oppenheim's knock-out options (source: Frankfurter Allgemeine Zeitung, November 2004)

Basiswert: DAX 4.130,81 +41,68 +1,02% 11.11.2004 Java-Applet: aktiv Neu Starten

WKN	Typ	Bid	Zeit	Ask	Zeit	Strike	StopLoss	Währung	BV	Fälligkeit
SAL60F	Long	2.470	7:05:14 PM	2.490	7:05:14 PM	3.900,00	3.900,00	XXP	0,01	23.12.2004
SAL6UC	Long	2.650	7:05:24 PM	2.670	7:05:24 PM	3.900,00	3.900,00	XXP	0,01	24.03.2005
SAL60G	Long	2.240	7:05:14 PM	2.260	7:05:14 PM	3.925,00	3.925,00	XXP	0,01	23.12.2004
SAL6U9	Long	1.970	7:05:14 PM	1.990	7:05:14 PM	3.950,00	3.950,00	XXP	0,01	23.12.2004
SAL6UA	Long	1.730	7:05:14 PM	1.750	7:05:14 PM	3.975,00	3.975,00	XXP	0,01	23.12.2004
SAL6UB	Long	1.470	7:05:14 PM	1.490	7:05:14 PM	4.000,00	4.000,00	XXP	0,01	23.12.2004
SAL4VM	Short	0.160	7:05:14 PM	0.180	7:05:14 PM	4.150,00	4.150,00	XXP	0,01	23.12.2004
SAL4VN	Short	0.410	7:05:14 PM	0.430	7:05:14 PM	4.175,00	4.175,00	XXP	0,01	23.12.2004
SAL1S6	Short	0.660	7:05:24 PM	0.670	7:05:24 PM	4.200,00	4.200,00	XXP	0,01	23.12.2004
SAL2GN	Short	0.610	7:05:27 PM	0.630	7:05:27 PM	4.200,00	4.200,00	XXP	0,01	24.03.2005

Figure 39: Bid-/Ask information of Sal. Oppenheim's knock-out options

Hedging exotic options

In BS world prices of barrier options are given analytically, all greeks can be calculated directly.
There exists static replication for some barrier option if:

- the underlying has no drift
- the IV on the market only depends on time not on strike

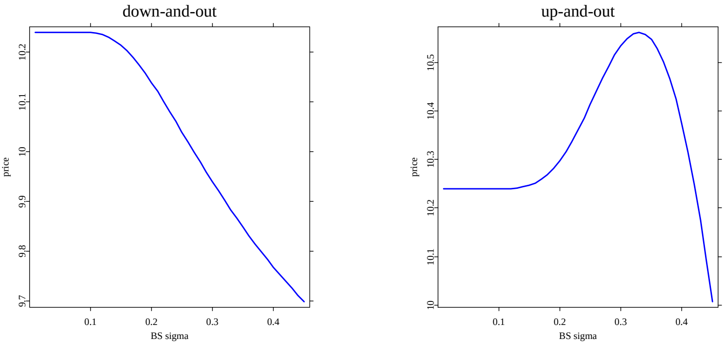


Figure 40: *Price of the call knock-out barrier options as a function of BS-σ. Asset value* $S_0 = 90$, *strike price* $K = 80$ *time to maturity* $\tau = 0.1$ *interest rate* $r = 0.03$. *Left panel: barrier* $B = 80$. *Right panel: barrier* $B = 120$.

Example

Consider a short position in a knock-out call option (C^{KO}) with strike 100 and barrier 90. Consider also one long position in a European call with strike 100 and a short position in 100/90 European puts with strike 81.

- if spot is at the barrier level 90 call and put would be worth the same
- if barrier was not reached before maturity the payoff of C^{KO} is equal to the payoff of the call

C^{KO} is replicated with vanilla options.

Position	Value at time t hits barrier	Value at time T doesn't hit barrier
C	$BS_{call}(K=100)$	$(S_T-100)^+$
$-100/90P$	$-\frac{100}{90}BS_{put}(K=81)$	0
$-C^{KO}$	0	$-(S_T-100)^+$
Sum	0	0

For each time t and each value of σ if $r=0$ and $S_t=90$ then $BS_{call}(K=100)=\frac{100}{90}BS_{put}(K=81)$

Dynamic hedging

Use approximation of the option value changes and adjust constantly the hedge portfolio.

$$\Delta C^{KO}(\Delta S, \Delta\sigma) \approx \frac{\partial C^{KO}}{\partial S}\Delta S + \frac{\partial C^{KO}}{\partial \sigma}\Delta\sigma$$

The changes in the asset price (delta risk) can be hedge the asset itself. The changes in volatility (vega risk) can be hedge with at-the-money plain vanilla call option (C).

Dynamic hedging

The sensitivity of the hedge portfolio $HP = a_1 S + a_2 C$ w.r.t. S and σ should be equal to the sensitivity of the C^{KO}. The hedge coefficients $a1, a2$ are given by the equation:

$$\begin{pmatrix} 1 & \frac{\partial C}{\partial S} \\ 0 & \frac{\partial C}{\partial \sigma} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C^{KO}}{\partial S} \\ \frac{\partial C^{KO}}{\partial \sigma} \end{pmatrix}$$

Local Volatility Model

In local volatility (LV) models the asset price dynamics are governed by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S_t, t) dW_t \tag{14}$$

where W_t is a Brownian motion, μ the drift and $\sigma(S_t, t)$ the local volatility function which depends on the asset price and time only.

For pricing the options the partial differential equation (18) is solved. Price depends on the **entire** IVS. From the IVS one can calculate $C_t(K, T)$.
Dupire formula:

$$\sigma^2(S_t, t) = 2\frac{\frac{\partial C_t(K,T)}{\partial T} + rK\frac{\partial C_t(K,T)}{\partial K}}{K^2\frac{\partial^2 C_t(K,T)}{\partial K^2}}$$

gives the local volatility surface $\sigma(S_t, t)$.

Hedging exotic options

Most greeks can be calculated:

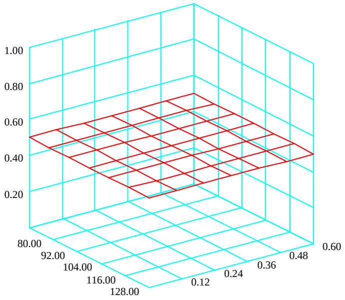
- Delta, $\frac{\partial C^{KO}}{\partial S}$, gamma, $\frac{\partial^2 C^{KO}}{\partial S^2}$ and theta, $\frac{\partial C^{KO}}{\partial t}$, can be read from the grid of the finite difference scheme;
- rho, $\frac{\partial C^{KO}}{\partial r}$, and dividend-rho, $\frac{\partial C^{KO}}{\partial \delta}$, are typically computed via a difference quotient assuming a flat term structure.

What about the vega ??
The usual vega, $\frac{\partial C^{KO}}{\partial \sigma}$ cannot be used since the entire IVS is input.

Classical vega hedging

Classical vega hedging corresponds to parallel move of IVS

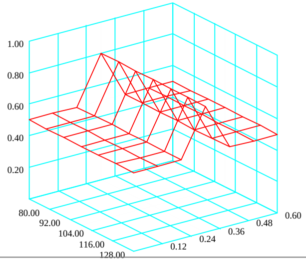
- In BS there is only one volatility number
- In LV it protects only of parallel move of the smile (β_1 effect)



Bucket hedging

With term structure of the IVS one may compute a bucket vega hedging. It provides a sensitivity measure of parallel movements over each maturity string.

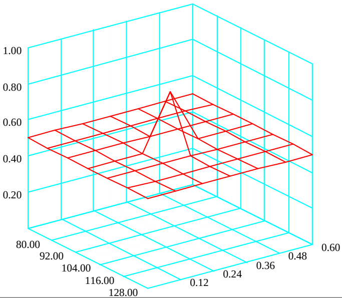
- The procedure indicates which European option maturities should be used for hedging
- Sensitivity related to strike is not given



Superbucket hedging

In superbucket analysis one has to compute sensitivity of exotics w.r.t. a move of each individual implied volatility.

- Sensitivity by strike and maturity is obtained
- The calculation needs to be done for each single point



Vega-hedging of the two DSFM factors

In DSFM the IV decomposition is given only by $L+1$ factors:

$$\widehat{\sigma}_i = \exp\left(\sum_{l=0}^{L} \widehat{\beta}_{i,l}\, \widehat{m}_l\right).$$

We can compute the sensitivities w.r.t. the factor loadings $\widehat{\beta}_l$!
From the interpretations, we receive an immediate understanding of the sensitivities:

- $\frac{\partial}{\partial \widehat{\beta}_1}$ is an **up-and-down shift vega** of the IVS;
- $\frac{\partial}{\partial \widehat{\beta}_3}$ is a **slope shift vega** of the IVS.

How to compute the hedge ratios

Take two hedge portfolios HP_1 and HP_2.
Compute the sensitivities of the hedge portfolios and the knock-out option with respect to $\widehat{\beta}_1$ and $\widehat{\beta}_3$.
Solve

$$\begin{pmatrix} \frac{\partial HP_1}{\partial \widehat{\beta}_1} & \frac{\partial HP_2}{\partial \widehat{\beta}_1} \\ \frac{\partial HP_1}{\partial \widehat{\beta}_3} & \frac{\partial HP_2}{\partial \widehat{\beta}_3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C^{KO}}{\partial \widehat{\beta}_1} \\ \frac{\partial C^{KO}}{\partial \widehat{\beta}_3} \end{pmatrix}$$

for the hedge ratios a_1, a_2.

Choice of the hedge portfolio

Idea:
choose HP_1 and HP_2 with *maximum exposure* to $\widehat{\beta}_1$ and $\widehat{\beta}_3$, respectively:

HP_1 should be most sensitive to up-and-down shifts:
use a portfolio of at-the-money plain vanilla options;

HP_2 should be most sensitive to slope changes:
use a portfolio of vega-neutral risk reversals.

Then $\frac{\partial HP_1}{\partial \widehat{\beta}_3} \approx 0$ and $\frac{\partial HP_2}{\partial \widehat{\beta}_1} \approx 0$.

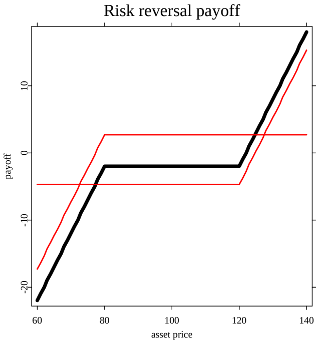


Figure 41: *The payoff of the risk reversal. It is compounded from long call with strike* $K_1 = 120$ *and short put with strike* $K_2 = 80$.

Outlook

Agenda:

- local-linear smoothing ✓
- data driven choice of L (number of m), and bandwidth h ✓
- forecasting exercise (almost done)
- investigate obvious relations to **Kalman Filtering**, Fengler et al. (2005):

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) + \epsilon_i \tag{15}$$

$$\beta_i = \widetilde{\beta}_i(\theta) + \eta_i \tag{16}$$

Outlook

Agenda:

- hedging empirical studies
- estimation of state price density (SPD)

$$f_{T-t}(K) = e^{r(T-t)} \frac{\partial^2 C_t(K,T)}{\partial K^2} \tag{17}$$

where $f_{T-t}(K)$ is SPD of the time T taken in the time t

Skew Hedging

Barrier options

Knock-out options are financial options that become worthless as soon as the underlying reaches a prespecified barrier.

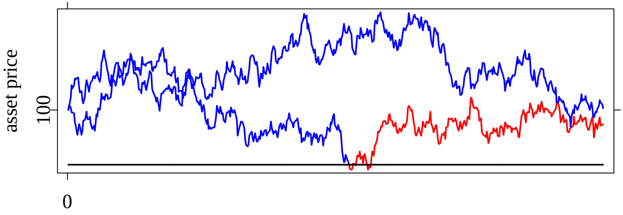


Figure 42: Example of two possible paths of asset's price. When the price hits the barrier (red) the option is no longer valid regardless further evolution of the price.

Barrier options

- In BS world prices of barrier options are given analytically, all greeks can be calculated directly.
- The price doesn't need to be a increasing function of the volatility parameter σ.
- Marking to BS model is precluded due to the σ choice
- BS is not a good choice for handling barrier options!!!

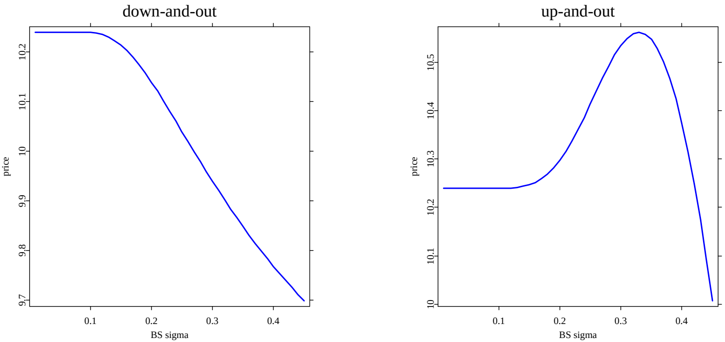


Figure 43: *Price of the call knock-out barrier options as a function of BS-σ. Asset value $S_0 = 90$, strike price $K = 80$ time to maturity $\tau = 0.1$ interest rate $r = 0.03$. Left panel: barrier $B = 80$. Right panel: barrier $B = 120$.*

Pricing Barrier Options

For pricing barrier options a local volatility (LV) model is employed. The asset price dynamics are governed by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S_t, t) dW_t \tag{18}$$

where W_t is a Brownian motion, μ the drift and $\sigma(S_t, t)$ the local volatility function which depends on the asset price and time only.

Pricing Barrier Options

Price depends on the **entire** implied volatility surface (IVS). From the IVS one can calculate $C_t(K, T)$.
Dupire formula:

$$\sigma^2(K,t) = 2\frac{\frac{\partial C_t(K,T)}{\partial T} + rK\frac{\partial C_t(K,T)}{\partial K}}{K^2\frac{\partial^2 C_t(K,T)}{\partial K^2}}$$

gives the LV surface $\sigma(S_t, t)$. For practical implementation see Andersen and Brotherthon-Ratcliffe (1997).

Dynamics of the IVS

The IVS reveals highly dynamic behavior, which influences the prices of the barrier options.

Example

Consider two one year knock-out put options with strike 110 and barrier 80, when the current spot level is 100. Take the IVS from 20000103 and 20010102. The prices of these options are respectively 1.91 and 2.37. This is a 25% difference.

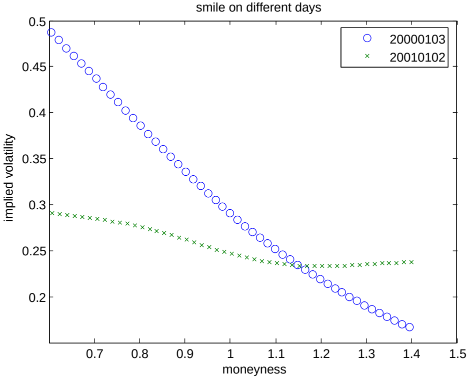


Figure 44: *Observed smile on 20000103 and 20010102 for the maturity 0.25.*

Vega Hedging

- In LV model the usual vega cannot be used because the whole IVS is an input
- The standard approach is to build vega hedging on the sensitivity of the “up-and-down” shifts.
- The skew changes, which may cause significant pricing differences, become unhedged.

DSFM

A complex dynamics of the IVS is explained in terms of a **dynamic semiparametric factor model (DSFM)** for the (log)-IVS
$Y_{i,j}$ ($i =$ day, $j =$ intraday):

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}. \tag{19}$$

Here $m_l(X_{i,j})$ are smooth factor functions and $\beta_{i,l}$ is a multivariate (loading) time-series.

Aims

- to apply DSFM for identification of key factors of the IVS dynamics
- to improve the vega hedge by hedging against most common changes of the IVS

Overview

1. Motivation✓
2. Dynamic Semiparametric Factor Model
3. Hedging Approach
4. Results
5. Conclusion

DSFM

Consider DSFM for the log-IVS:

$$Y_{i,j} = m_0(X_{i,j}) + \sum_{l=1}^{L} \beta_{i,l} m_l(X_{i,j}) + \varepsilon_{i,j}, \tag{20}$$

$Y_{i,j}$ is log IV, i denotes the trading day $(i = 1, \ldots, I)$,
$j = 1, \ldots, J_i$ is an index of the traded options on day i.
$m_l(\cdot)$ for $l = 0, \ldots, L$ are basis functions in covariables $X_{i,j}$ (moneyness, time to maturity),
and β_i are time dependent factors.

DSFM estimation

Define estimates of $\widehat{m}_l$ and $\widehat{\beta}_{i,l}$ with $\widehat{\beta}_{i,0} \stackrel{\mathrm{def}}{=} 1$, as minimizers of:

$$\sum_{i=1}^{I} \sum_{j=1}^{J_i} \int \left\{ Y_{i,j} - \sum_{l=0}^{L} \widehat{\beta}_{i,l} \widehat{m}_l(u) \right\}^2 K_h(u - X_{i,j}) \, du, \qquad (21)$$

where K_h denotes a two dimensional product kernel,
$K_h(u) = k_{h_1}(u_1) \times k_{h_2}(u_2), \quad h = (h_1, h_2)$ with a one-dimensional kernel $k_h(v) = h^{-1}k(h^{-1}v)$.
See Fengler et al. (2005), Fengler (2005).

Model parameters

We fit our model:

- $L = 3$ dynamic basis functions
- grid covering moneyness $\in [0.6, 1.3]$ and time to maturity $\in [0.05, 1]$
- fix bandwidths in moneyness direction and increasing bandwidths in maturity direction
- on the daily IVS data from 20000103 till 20011220

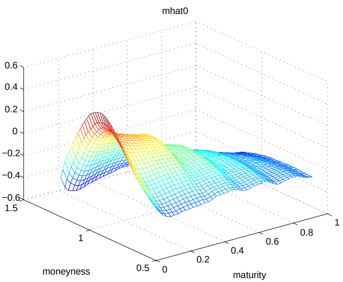


Figure 45: *Invariant basis function* $\widehat{m}_0$ *and dynamic basis function* $\widehat{m}_1$ *(level)*

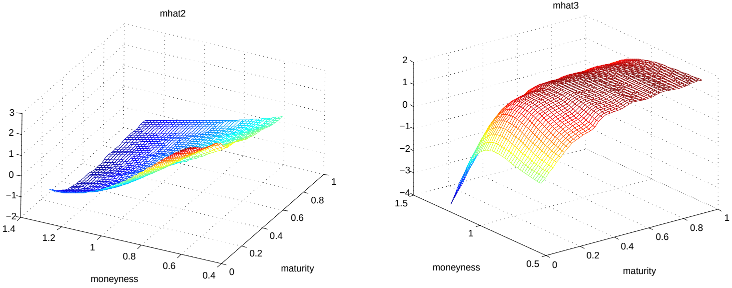


Figure 46: *Dynamic basis functions* $\widehat{m}_2$ *(skew) and* $\widehat{m}_3$ *(term structure)*

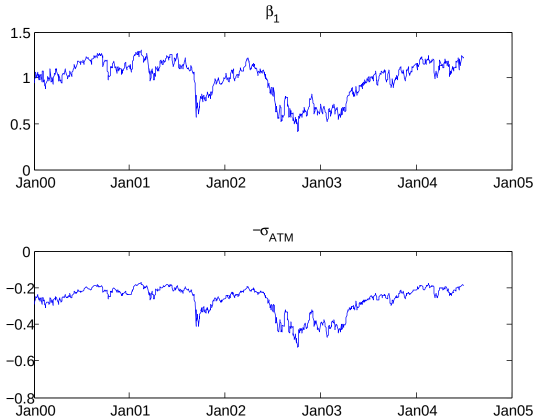


Figure 47: *time series of weights $\widehat{\beta}_1$ and ATM IVS for the fixed maturity 0.25.*

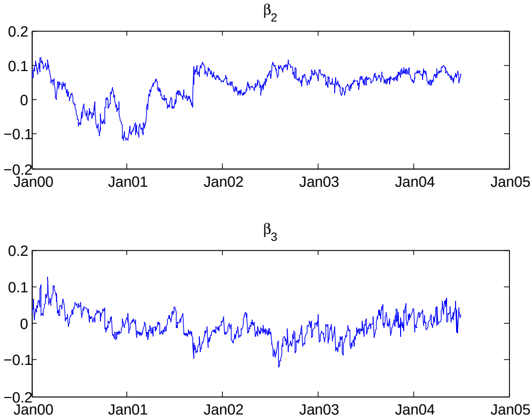


Figure 48: *Time series of weights* $\widehat{\beta}_2$ *and* $\widehat{\beta}_3$

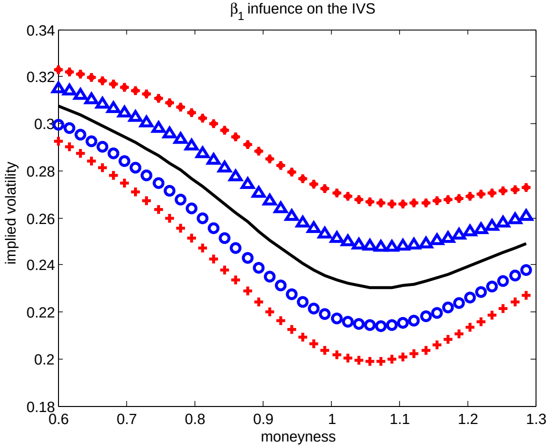


Figure 49: *Typical shape of the smile for different levels of* $\widehat{\beta}_1$. *Changes of the* $\widehat{\beta}_1$ *influence mainly the surface's level.*

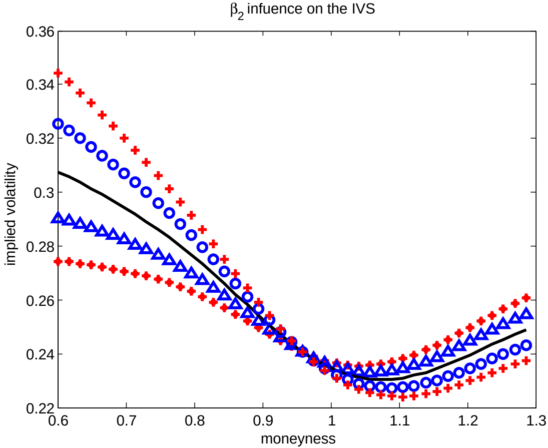


Figure 50: *Typical shape of the smile for different levels of $\widehat{\beta}_2$. Changes of the $\widehat{\beta}_2$ influence the smile's skew.*

Greeks

- In order to implement β-hedging one has to calculate β-greeks.
- They are obtained by shifting the IVS in the $\widehat{m}$ direction.

$$\frac{\partial option}{\partial \widehat{\beta}} \approx \frac{option(IVSe^{\Delta\widehat{\beta}\widehat{m}}) - option(IVSe^{-\Delta\widehat{\beta}\widehat{m}})}{2\Delta\widehat{\beta}} \qquad (22)$$

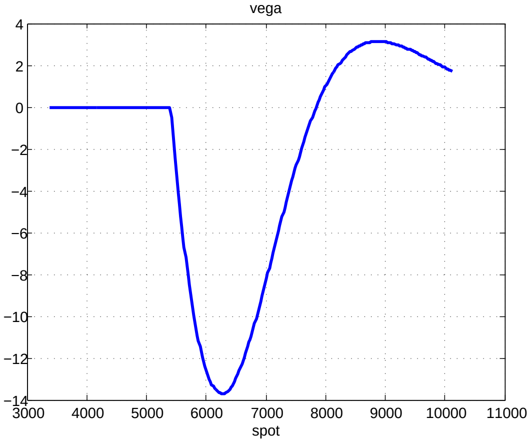


Figure 51: *vega "greek" for down-and-out put option with barrier* 5400 *and strike* 7425 *as a function of spot*

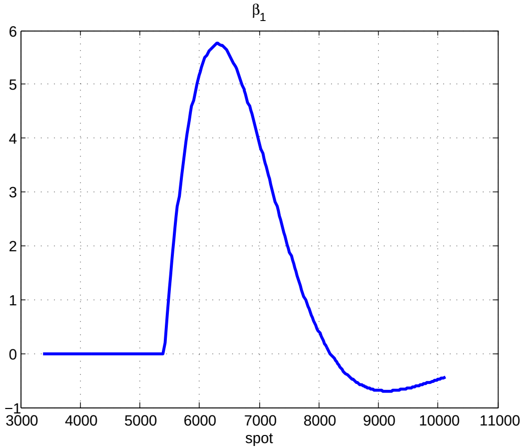


Figure 52: $\widehat{\beta}_1$ *"greek" for down-and-out put option with barrier* 5400 *and strike* 7425 *as a function of spot*

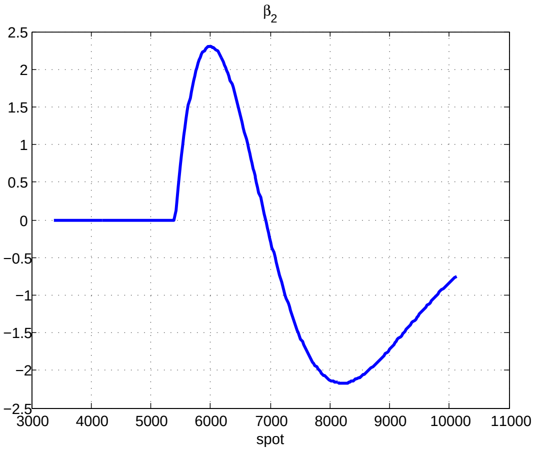


Figure 53: $\widehat{\beta}_2$ *"greek" for down-and-out put option with barrier* 5400 *and strike* 7425 *as a function of spot*

Example

In the BS model the hedge portfolio (HP) for hedging plain vanilla options consists of a stocks - $HP = aS$. The hedge ratio a (delta) is obtained from:

$$\frac{dHP}{dS} = a = \frac{\partial option}{\partial S}.$$

The hedge is financed by buying/selling bonds.

How to compute the hedge ratios

Take two hedge portfolios HP_1 and HP_2.
Compute the sensitivities of the hedge portfolios and the up-and-out call option (C^{KO}) with respect to $\widehat{\beta}_1$ and $\widehat{\beta}_2$.
Solve

$$\begin{pmatrix} \frac{\partial HP_1}{\partial \widehat{\beta}_1} & \frac{\partial HP_2}{\partial \widehat{\beta}_1} \\ \frac{\partial HP_1}{\partial \widehat{\beta}_2} & \frac{\partial HP_2}{\partial \widehat{\beta}_2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial C^{KO}}{\partial \widehat{\beta}_1} \\ \frac{\partial C^{KO}}{\partial \widehat{\beta}_2} \end{pmatrix} \begin{pmatrix} vega \\ skew \end{pmatrix}$$

for the hedge ratios a_1, a_2. For the down-and-out put option (P^{KO}) the procedure is analogous.

Choice of the hedge portfolio

Idea:
choose HP_1 and HP_2 with *maximum exposure* to $\widehat{\beta}_1$ and $\widehat{\beta}_2$, respectively:

HP_1 should be most sensitive to up-and-down shifts:
use a portfolio of at-the-money plain vanilla options;

HP_2 should be most sensitive to slope changes:
use a portfolio of vega-neutral risk reversals.

Then $\frac{\partial HP_1}{\partial \widehat{\beta}_2} \approx 0$ and $\frac{\partial HP_2}{\partial \widehat{\beta}_1} \approx 0$.

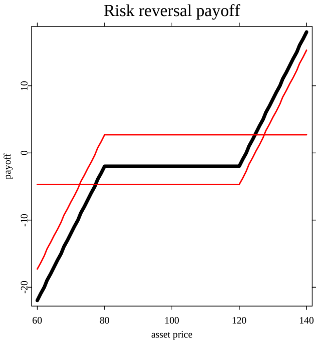
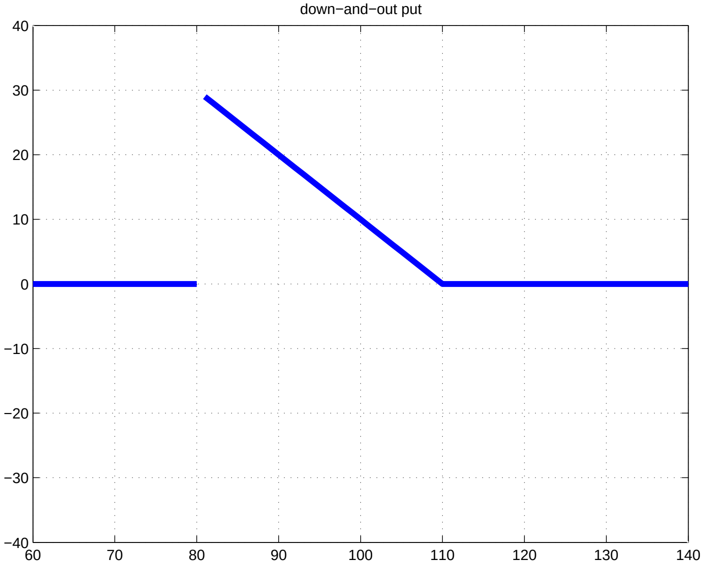


Figure 54: *The payoff of the risk reversal. It is composed from a long call with strike* $K_1 = 120$ *and a short put with strike* $K_2 = 80$*.*

As in standard vega hedging we apply final delta hedge. In our case we apply delta hedge to $C^{KO} + a_1 HP_1 + a_2 HP_2$ by calculating the number of underlying as:

$$\frac{\partial (C^{KO} + a_1 HP_1 + a_2 HP_2)}{\partial S}$$


down−and−out put
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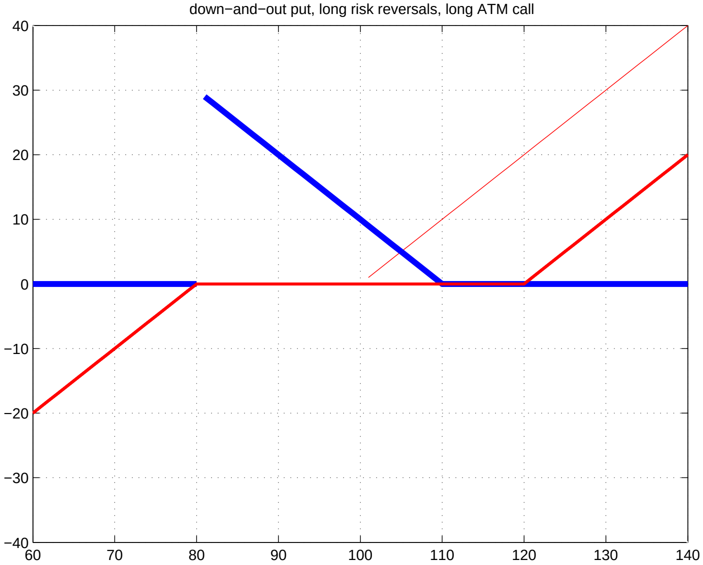
down−and−out put, long risk reversals, long ATM call
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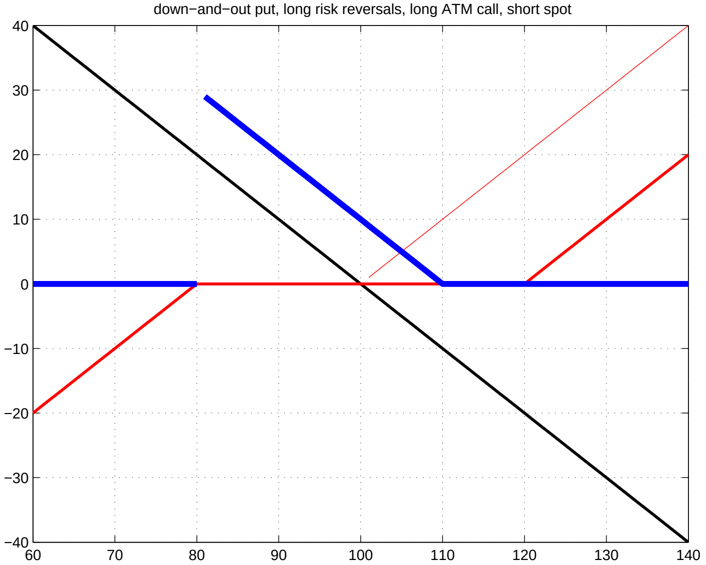
down−and−out put, long risk reversals, long ATM call, short spot
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Empirical Study

For each of 885 days (20000103-20030707) we start one long position in one year C^{KO} and P^{KO}.

Option	barrier	strike	maturity	knock-outs	in-the-money
C^{KO}	140 %	80 %	1 year	10 %	39 %
P^{KO}	80 %	110 %	1 year	81 %	5 %

Table 15: *barrier and strike are given as a percentage of the spot at the starting day*

We keep the position until maturity or knock-out.

Empirical Study

We compare the $\widehat{\beta}_1\widehat{\beta}_2$ (skew) hedging approach with:

- $\widehat{\beta}_1$ hedging - no risk reversal $(a_2 = 0)$ and $a_1 = \frac{-\partial C^{KO}}{\partial \widehat{\beta}_1} / \frac{\partial HP_1}{\partial \widehat{\beta}_1}$
- vega hedging - no risk reversal $(a_2 = 0)$ and $a_1 = \frac{-\partial C^{KO}}{\partial \sigma} / \frac{\partial HP_1}{\partial \sigma}$

Aims of Hedging

- We define the profit and loss of the strategy at the maturity as a portfolio's value divided by notional at the starting day.

$$\frac{C_T^{KO} + HP_T + money_T}{S_0}$$

- The aim of the hedging is possibly large reduction of the profit and loss variation around zero.

Results

Profit and loss of the strategy at the maturity.

C^{KO}	min	max	mean	median	std	med. abs.
vega	-0.1038	0.5813	-0.0165	-0.0175	0.0209	0.0413
β_1	-0.0752	0.5768	-0.0118	-0.0136	0.0183	0.0387
$\beta_1\beta_2$	-0.0830	0.5684	-0.0066	-0.0119	0.0137	0.0345

Table 16: *all values as percentage of the underlying*

C^{KO}	days	min	max	mean	median	std	med. abs.
vega	0	-0.1038	0.5813	-0.0165	-0.0175	0.0209	0.0413
	1	-0.1038	0.1710	-0.0186	-0.0171	0.0183	0.0276
	10	-0.0833	0.0710	-0.0184	-0.0164	0.0172	0.0241
	25	-0.0797	0.0590	-0.0191	-0.0151	0.0150	0.0207
β_1	0	-0.0752	0.5768	-0.0118	-0.0136	0.0183	0.0387
	1	-0.0751	0.1459	-0.0139	-0.0130	0.0157	0.0240
	10	-0.0766	0.0702	-0.0143	-0.0130	0.0154	0.0210
	25	-0.0731	0.0508	-0.0150	-0.0116	0.0130	0.0175
$\beta_1\beta_2$	0	-0.0830	0.5684	-0.0066	-0.0119	0.0137	0.0345
	1	-0.0829	0.1220	-0.0088	-0.0120	0.0112	0.0184
	10	-0.0375	0.0831	-0.0095	-0.0119	0.0106	0.0149
	25	-0.0360	0.0499	-0.0104	-0.0123	0.0082	0.0114

Table 17: Descriptive statistics for the hedging strategies 0, 1, 10 and 25 days before the knock-out or expiration - delta hedging effect (gap risk).

Results

Profit and loss of the strategy at the maturity.

P^{KO}	min	max	mean	median	std	med. abs.
vega	-0.0264	0.2799	0.0058	-0.0004	0.0105	0.0213
β_1	-0.0210	0.2808	0.0080	0.0016	0.0107	0.0214
$\beta_1\beta_2$	-0.0332	0.2676	0.0065	0.0008	0.0092	0.0196

Table 18: Descriptive statistics for the hedging strategies of the down-and-out put

P^{KO}	days	min	max	mean	median	std	med. abs.
vega	0	-0.0264	0.2799	0.0058	-0.0004	0.0105	0.0213
	1	-0.0209	0.0344	-0.0040	-0.0048	0.0042	0.0064
	10	-0.0161	0.0231	-0.0024	-0.0027	0.0037	0.0056
	25	-0.0142	0.0189	-0.0018	-0.0014	0.0033	0.0046
β_1	0	-0.0210	0.2808	0.0080	0.0016	0.0107	0.0214
	1	-0.0157	0.0350	-0.0017	-0.0030	0.0038	0.0060
	10	-0.0106	0.0276	-0.0002	-0.0009	0.0031	0.0053
	25	-0.0109	0.0202	-0.0001	-0.0002	0.0027	0.0041
$\beta_1\beta_2$	0	-0.0332	0.2676	0.0065	0.0008	0.0092	0.0196
	1	-0.0249	0.0270	-0.0032	-0.0032	0.0030	0.0044
	10	-0.0110	0.0200	-0.0017	-0.0016	0.0027	0.0038
	25	-0.0092	0.0200	-0.0011	-0.0007	0.0023	0.0034

Table 19: Descriptive statistics for the hedging strategies 0, 1, 10 and 25 days before the knock-out or expiration - delta hedging effect (gap risk).

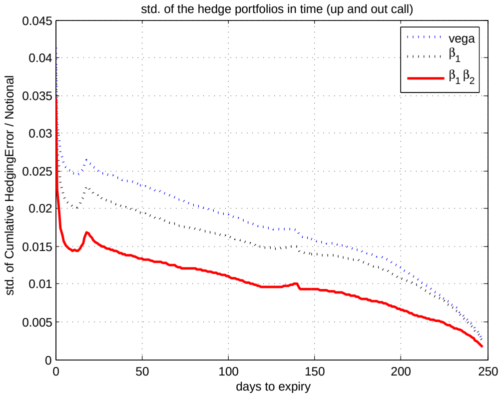

Figure 55: The standard deviation of the portfolios as a function of the days left to the maturity (call).

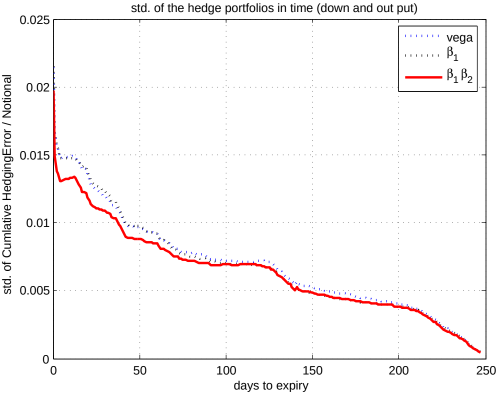

Figure 56: The standard deviation of the portfolios as a function of the days left to the maturity (put).

Conclusion

- the β hedge improves the hedging
- gap risk is still unhedged.
- better strategy might be to mix static and dynamic hedges

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