Measuring Statistical Risk
Extremes, joint extremes and copulae

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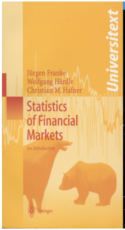
Risk Management
Risikomanagement
风险管理
危険と管理

ادارة المخاطرة

위험관리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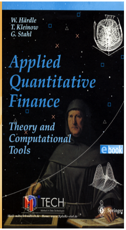
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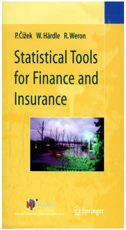
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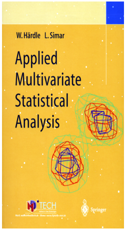
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Wertpapiergattungen
im AHW Top-Dividend Low-5 International

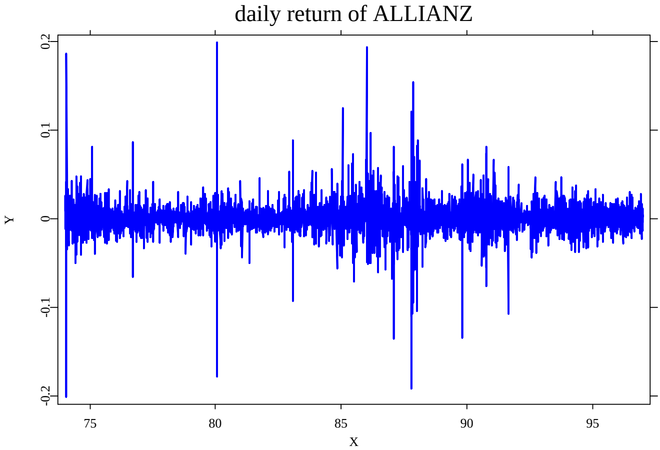
Aktie	Index		Dividendenrendite p.a.
ThyssenKrupp	DAX	3,86 %	3,82 %
TUI	DAX	4,53 %	3,76 %
Volkswagen AG	DAX	3,99 %	2,91 %
Deutsche Telekom	DAX	4,15 %	4,01 %
Daimler Chrysler AG	DAX	4,18 %	4,36 %
MAN AG	DAX	4,16 %	3,08 %
Credit Agricole	CAC-40	3,80 %	3,10 %
DEXIA SA	CAC-40	4,25 %	3,42 %
AGF	CAC-40	4,46 %	4,31 %
AXA	CAC-40	4,22 %	2,96 %
Arcelor	CAC-40	3,87 %	3,72 %
BNP	CAC-40	4,22 %	3,64 %
ABN Amro	AEX	4,09 %	5,19 %
Ing Groep NV	AEX	4,23 %	4,59 %
Fortis	AEX	4,37 %	7,11 %
Reed Elsevier NV	AEX	3,03 %	2,86 %
Aegon	AEX	4,00 %	4,06 %
Wolters Kluwer	AEX	3,51 %	3,90 %
KPN	AEX	4,17 %	5,07 %
General Electric	Dow-Jones	3,75 %	2,47 %
JP Morgan Chase	Dow-Jones	3,54 %	3,93 %
Verizon Communications	Dow-Jones	4,19 %	4,65 %
Merck & Co INC	Dow-Jones	4,30 %	4,75 %
Pfizer INC	Dow-Jones	4,06 %	2,95 %
SBC Communications INC	Dow-Jones	4,06 %	5,53 %
Barreserve		- 0,99 %	

Aktueller Zinssatz der Europäischen Zentralbank: **2,00 %**

Stand: 30.03.2005
Quellen: Dt. Bundesbank, Thomson Financial Datastream, Bloomberg.

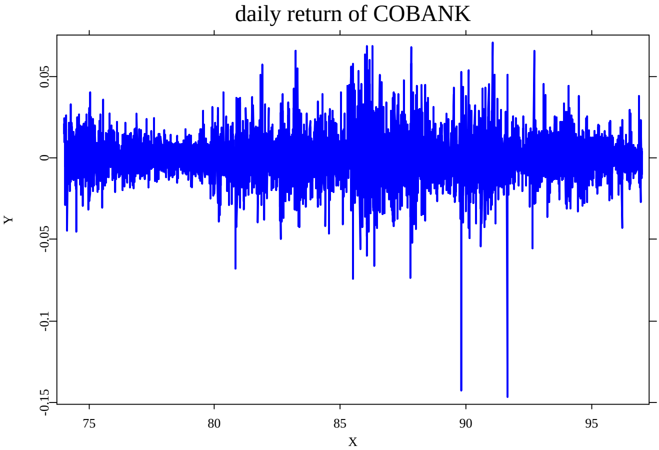
Example

Daily returns of the German stock Allianz from 1974-01-02 to 1996-12-30.



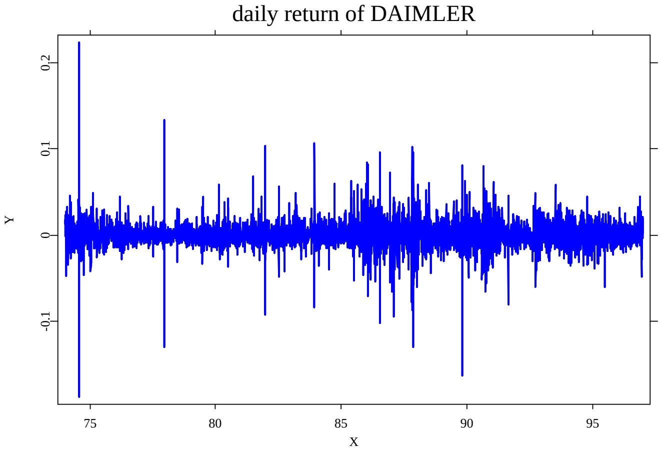
Example

Daily returns of the German stock COBANK from 1974-01-02 to 1996-12-30.



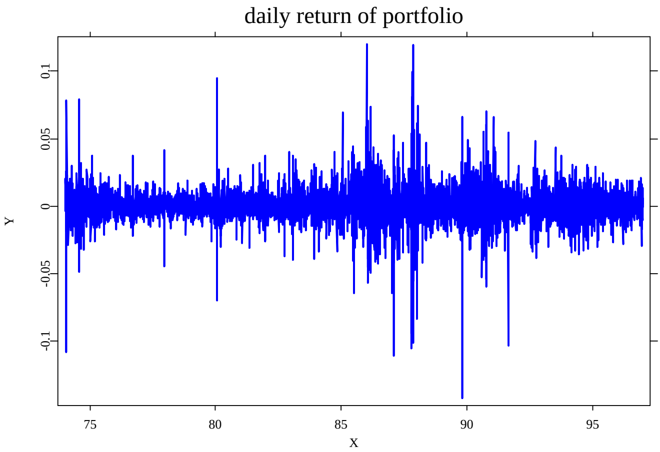
Example

Daily returns of the German stock DAIMLER from 1974-01-02 to 1996-12-30.



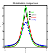
Example

Daily returns of the German stock portfolio (ALLIANZ, COBANK and DAIMLER) with trading strategy $b^\top = (1, 1, 1)$ from 1974-01-02 to 1996-12-30.



Outline of the talk

1. Motivation ✓
2. Extreme Values
3. Copulae
4. Tail Dependence



Extreme Value

Extremwert

极值

極値

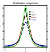
القيمة الحدية

극단값

Statistics of Extreme Risks

Stylized facts in financial markets

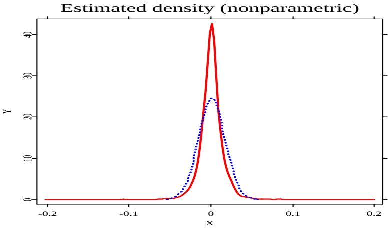
- Returns are heavy tailed distributed
- Volatility changes stochastically
- GARCH model yields fat tails but often underestimates for $q \geq 95\%$.

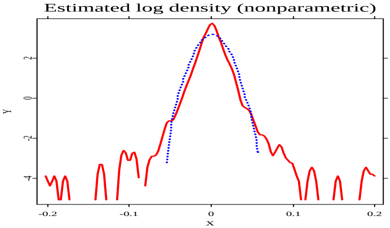


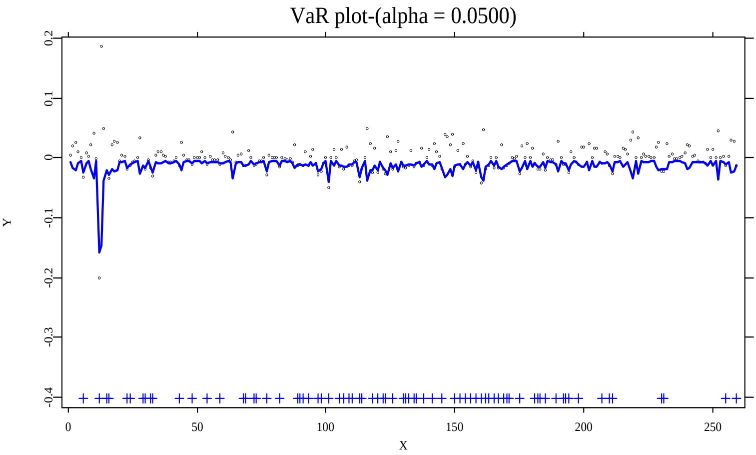
Example

Model structure of the return series of Allianz from 1974-01-02 to 1996-12-30:

$$X_t = \sigma_t \varepsilon_t, \quad \varepsilon_t \sim \ iid(0,1) \tag{1}$$
$$\sigma_t^2 = \omega + \alpha_1 X_{t-1}^2 + \beta \sigma_{t-1}^2 \tag{2}$$

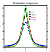





VaR plot-(alpha = 0.0500)
Y
X
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
50
100
150
200
250

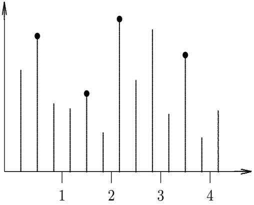
Extreme value distributions

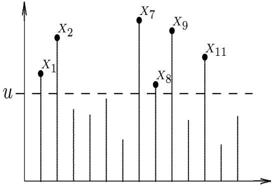
- yield more precise approximations in the tails
- probability of extreme events depends on the tail of $f(x)$ - pdf of ε_t
- apply methods of extreme value statistics to estimate "extreme" quantiles



Identifying extreme events

- Maxima (block maxima) taking in successive periods
- Peaks over threshold (POT): loss exceeds a given (high) threshold u.





The limits of maxima

Let $X_1, \ldots, X_n$ be iid random variables (P & L) with cdf $F(x)$

$$M_n = \max(X_1, \ldots, X_n)$$

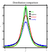
One may easily compute the cdf of maxima:

$$\mathrm{P}(M_n \leq x) = \mathrm{P}(X_1 \leq x, \ldots, X_n \leq x) = F^n(x). \tag{3}$$

For unbounded random variables, (i.e. $F(x) < 1,\ \forall x < \infty$):

$$F^n(x) \to 0, \quad \text{hence} \quad M_n \xrightarrow{\mathrm{P}} \infty$$

The maximum of n unbounded random variables may become arbitrarily large.



Definition (Maximum Domain of Attraction)

The random variables X_t belong to the maximum domain of attraction (MDA) of the nondegenerated distribution G, if there exist constants $c_n > 0$ and d_n such that:

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G \quad \text{for} \quad n \to \infty,$$

i.e. $F^n(c_n x + d_n) \to G(x)$ for all points of continuity x of the cdf $G(x)$.

Remark: Extreme value distribution

Distribution G in the above Definition is called an *extreme value (EV) distribution*.

Three standard extreme value distributions:

$$\begin{aligned} \textit{Fréchet:} \quad G_{1,\alpha}(x) &= \exp\{-x^{-\alpha}\}, \; x \geq 0 \;, \alpha > 0, && (4) \\ \textit{Gumbel:} \quad G_0(x) &= \exp\{-e^{-x}\}, \; x \in \mathbb{R}, && (5) \\ \textit{Weibull:} \quad G_{2,\alpha}(x) &= \exp\{-|x|^{-\alpha}\}, \; x \leq 0 \;, \alpha < 0. && (6) \end{aligned}$$

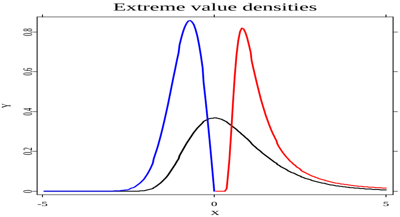
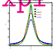


Figure 1: Fréchet (red), Gumbel (black) and Weibull (blue) probability density functions.

SFEevt1.xpl



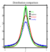
Jenkinson and von Mises suggested a parametric representation for the three standard distributions:

Definition (Generalized Extreme Value)

The generalized extreme value distribution (GEV) with the shape parameter $\gamma \in \mathbb{R}$ has the following cdf:

$$G_\gamma(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\},\ 1+\gamma x > 0 \ \text{ for } \gamma \neq 0$$
$$G_0(x) = \exp\{-e^{-x}\},\ x \in \mathbb{R}$$

Gumbel	G_0
Fréchet	$G_\gamma(\frac{x-1}{\gamma}) = G_{1,1/\gamma}(x)$ for $\gamma > 0$
Weibull	$G_\gamma(-\frac{x+1}{\gamma}) = G_{2,-1/\gamma}(x)$ for $\gamma < 0$.



Richard Edler von Mises
born on April 19, 1883 in Lviv, Austria-Hungary
died on July 14, 1953 in Boston, USA

Figure 2:

Richard von Mises was a scientist who worked on fluid mechanics, aerodynamics, aeronautics, statistics and probability theory. After World War I in 1919 he was appointed director (with full professorship) of the new Institute of Applied Mathematics created at the behest of Erhard Schmidt at the University of Berlin. With the rise of the National Socialist (Nazi) party to power in 1933, von Mises, who was a Roman Catholic but had Jewish ancestry, felt his position threatened despite his World War I military service. He moved to Turkey and to USA.

Theorem (Fisher and Tippett (1928) Theorem)

If there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non-degenerated distribution function G such that

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G \quad \text{for } n \to \infty,$$

then G is a GEV distribution.

Assume that we have a large enough block of n iid random variables and set $y = c_n x + d_n$, then $\mathrm{P}(M_n \leq y) \approx G_\gamma(\frac{y-d_n}{c_n})$.

$$F^{[nt]}(c_{[nt]}x + d_{[nt]}) \longrightarrow G(x) \ \text{ for } \ [nt] \to \infty, \quad \text{i.e.} \quad n \to \infty.$$

$$F^{[nt]}(c_n x + d_n) = \{F^n(c_n x + d_n)\}^{\frac{[nt]}{n}} \longrightarrow G^t(x) \text{ for } n \to \infty.$$

In other words this means that

$$\frac{M_{[nt]} - d_{[nt]}}{c_{[nt]}} \xrightarrow{\mathcal{L}} G \ , \qquad \frac{M_{[nt]} - d_n}{c_n} \xrightarrow{\mathcal{L}} G^t$$

for $n \to \infty$. According to the next lemma,

$$\frac{c_n}{c_{[nt]}} \longrightarrow b(t) \geq 0, \qquad \frac{d_n - d_{[nt]}}{c_{[nt]}} \longrightarrow a(t)$$

and

$$G^t(x) = G(b(t)x + a(t)), \ t > 0, \ x \in \mathbb{R}. \tag{7}$$

This relationship holds for arbitrary values t. We use it in particular for arbitrary t, s and $s \cdot t$ and obtain

$$b(st) = b(s)\ b(t),\ \ a(st) = b(t)a(s) + a(t). \tag{8}$$

The functional equations (7), (8) for $G(x), b(t), a(t)$ have only one solution, when G is one of the distributions $G_0, G_{1,\alpha}$ or $G_{2,\alpha}$, that is, G must be a GEV distribution.

Lemma (Convergence-Type Theorem)

Let $U_1, U_2, \ldots, V, W$ be random variables, $b_n, \beta_n > 0$, $a_n, \alpha_n \in \mathbb{R}$. If

$$\frac{U_n - a_n}{b_n} \xrightarrow{\mathcal{L}} V$$

in distribution for $n \to \infty$, then the following statement holds:

$$\frac{U_n - \alpha_n}{\beta_n} \xrightarrow{\mathcal{L}} W \qquad \textit{iff} \qquad \frac{b_n}{\beta_n} \longrightarrow b \geq 0, \; \frac{a_n - \alpha_n}{\beta_n} \longrightarrow a \in \mathbb{R}.$$

In this case W follows the same distribution as $bV + a$.

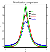
Notice that for all $n \geq 1$ the maximum M_n of n iid random variables $X_1, ..., X_n$ has the same distribution as $c_n X_1 + d_n$ given suitable constants $c_n > 0$ and d_n.

Properties of GEV

- In general we can change the center and the scale to obtain other GEV distributions:

$$G(x) = G_\gamma\left(\frac{x-\mu}{\sigma}\right)$$

with the shape parameter γ, the location parameter μ and the scale parameter $\sigma > 0$.



Properties of GEV

- GEV distributions are characterized by their max-stability. A probability density function F is max-stable if

$$F^n(d_n + c_n x) = F(x)$$

for a suitable choice of constants d_n and $c_n > 0$. For example, the maximum M_n of n iid random variables X_i has the same distribution as $c_n X_1 + d_n$ given suitable constants $c_n > 0$ and d_n.

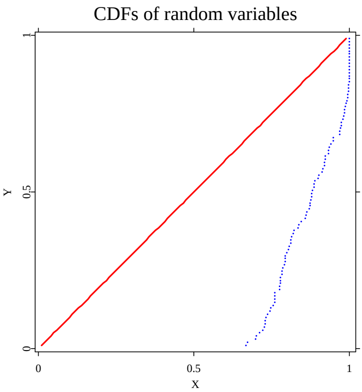


Figure 3: Normal PP plot of the pseudo random variables with Frechét distribution (4) with $\alpha = 2$. SFEevt2.xpl

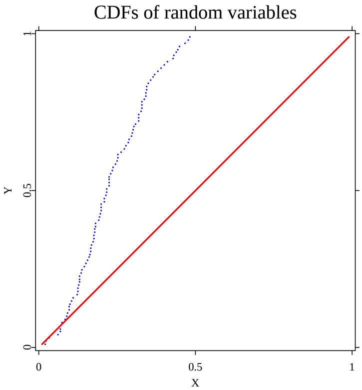


Figure 4: Normal PP plot of the pseudo random variables with Weibull distribution (6) with $\alpha = -2$. SFEevt2.xpl

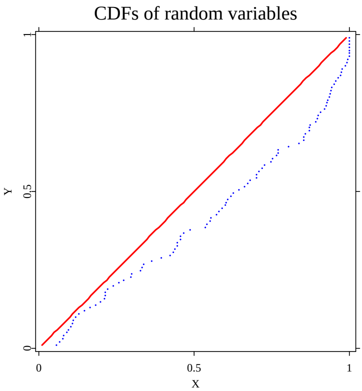


Figure 5: Normal PP plot of the pseudo random variables with Gumbel distribution (5). SFEevt2.xpl

Identifying the type of the limit (GEV) distributions

The deciding factor is how fast the probability for extremely large observations decreases beyond a threshold x, when x increases.
It depends obviously on the decrease of the function:

$$\bar{F}(x) = \mathrm{P}(X > x) = 1 - F(x)$$

for large x.

Theorem

a) For $0 \leq \tau \leq \infty$ *and every sequence of real numbers* $u_n, n \geq 1$, *it holds for* $n \to \infty$

$$n\bar{F}(u_n) \to \tau \quad \textit{iff} \quad \mathrm{P}(M_n \leq u_n) \to e^{-\tau}.$$

b) F *belongs to the MDA of the GEV distribution* G *with the standardized sequences* c_n, d_n, *exactly when* $n \to \infty$

$$n\bar{F}(c_n x + d_n) \to -\log G(x) \quad \textit{for all } x \in \mathbb{R}.$$

The excess probability of Fréchet $G_{1,\alpha}$ behaves as:

$$\bar{G}_{1,\alpha}(x) = \frac{1}{x^{\alpha}}\{1 + \mathcal{O}(1)\} \qquad \text{for} \quad x \to \infty.$$

All distributions that belong to the MDA of Fréchet $G_{1,\alpha}$ fulfill: $x^{\alpha}\bar{F}(x)$ is almost constant for $x \to \infty$ or more precisely $x^{\alpha}\bar{F}(x)$ is a slowly varying function.

Definition (Slowly Varying Functions)

A positive measurable function L in $(0, \infty)$ is called slowly varying, if for all $t > 0$:

$$\frac{L(tx)}{L(x)} \to 1 \qquad \text{for} \quad x \to \infty.$$

Example

$L(x) = \log(1 + x), x > 0$ is slowly varying (L'Hospital's rule).

Theorem (MDA of Frechét distribution)

F belongs to the MDA of the Frechét distribution $G_{1,\alpha}$, for $\alpha > 0$, if and only if $x^{\alpha}\bar{F}(x) = L(x)$ is a slowly varying function. The random variables X_t with the distribution function F are unbounded (i.e. $F(x) < 1$ for all $x < \infty$) and

$$\frac{M_n}{c_n} \xrightarrow{\mathcal{L}} G_{1,\alpha}$$

with $c_n = F^{-1}(1 - \frac{1}{n})$ or $\bar{F}(c_n) = \mathrm{P}(X_t > c_n) = 1/n$.

Theorem (MDA of Frechét distribution) states a criterion for obtaining the GEV Fréchet $G_{1,\alpha}$ as limit distribution.

The Weibull distribution can be obtained via the relationship $G_{2,\alpha}(-x^{-1}) = G_{1,\alpha}(x),\ x > 0,$. However random variables, whose maxima are asymptotically Weibull distributed, are by all means bounded. Therefore, in financial applications they are only interesting in special situations where using a type of hedging strategy, the loss, which results from an investment, is limited.

Example

The Pareto distributions with cdf

$$W_{1,\alpha}(x) = 1 - \frac{1}{x^{\alpha}}, x \geq 1, \alpha > 0,$$

and all other cdfs with Pareto tails:

$$\bar{F}(x) = \frac{\kappa}{x^{\alpha}}\{1 + \mathcal{O}(1)\} \quad \text{for} \quad x \to \infty.$$

belong to the MDA of the Fréchet distribution.
In this case $\bar{F}^{-1}(q)$ for $q \approx 1$ behaves as $(\kappa/q)^{1/\alpha}$: Set $c_n = (\kappa n)^{1/\alpha}$:

$$\frac{M_n}{(\kappa n)^{1/\alpha}} \xrightarrow{\mathcal{L}} G_{1,\alpha} \quad \text{for } n \to \infty$$

Theorem (MDA of Gumbel distribution)

The cdf F of X_t belongs to the MDA of the Gumbel distribution iff there exist scaling functions $c(x), g(x) > 0$ and an absolute cts function $e(x) > 0$:
$c(x) \to c > 0,\ g(x) \to 1, e'(x) \to 0$ *for* $x \to \infty$ *s.t.* $z < \infty$*:*
$\bar{F}(x) = c(x) \exp\{-\int_z^x \frac{g(y)}{e(y)} dy\},\ z < x < \infty$. *In this case*

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G_0$$

where $d_n = F^{-1}(1 - \frac{1}{n})$ *and* $c_n = e(d_n)$.

As function $e(x)$ in Theorem (MDA of Gumbel distribution) one may choose the *mean excess function:*

$$e(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(y)\, dy,\ \ x < \infty.$$

Example

The exponential distribution has the form: $F(x) = 1 - e^{-\lambda x}, x \geq 0$. Hence $\bar{F}(x) = e^{-\lambda x}$ fulfills the assumptions of Theorem (MDA of Gumbel distribution) with

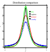
$$c(x) = 1,\ g(x) = 1,\ z = 0 \text{ and } e(x) = 1/\lambda.$$

Example

The maximum of n iid exponentially distributed random variables with the parameter λ converges to the GEV Gumbel distribution:

$$\lambda(M_n - \frac{1}{\lambda}\ \log\ n) \xrightarrow{\mathcal{L}} G_0$$

for $\quad n \to \infty$.



Example

The maximum of n iid $\mathrm{N}(0,1)$ random variables converges to the GEV Gumbel distribution:

$$\frac{M_n - d_n}{c_n} \xrightarrow{\mathcal{L}} G_0 \quad \text{for} \quad n \to \infty$$

where

$$\begin{aligned} c_n &= (2 \ \log \ n)^{-1/2} \\ d_n &= \sqrt{2 \ \log \ n} - \frac{\log\log \ n + \log(4\pi)}{2\sqrt{2 \ \log \ n}}. \end{aligned}$$

Peaks-over-threshold (POT) approach

Definition (Excess over threshold)

Let $K_n(u)$ and $N(u)$ be the index set and the number of observations over the threshold u. Denote the random variables $Y_l,\ l = 1, \ldots, N(u)$, as the excesses over the threshold value u with

$$
\begin{aligned}
\{Y_1, \ldots, Y_{N(u)}\} &= \{X_j - u;\ j \in K_n(u)\} \\
&= \{X^{(1)} - u, \ldots, X^{(N(u))} - u\}
\end{aligned}
$$

Definition

Let u be a threshold value and F a distribution function of some unbounded random variable X.

a) $F_u(x) = \mathrm{P}\{X - u \le x \mid X > u\} = \{F(u+x) - F(u)\}/\bar{F}(u),\ 0 \le x < \infty$ is called *conditional excess distribution function* over the threshold u.

b) $e(u) = \mathrm{E}\{X - u \mid X > u\},\ 0 < u < \infty$ is the mean excess function.

With partial integration one obtains:

$$e(u) = \int_u^{\infty} \frac{\bar{F}(y)}{\bar{F}(u)} dy.$$

A random variable Δ_u with cdf $F_u(x)$ has expected value $\mathrm{E}\,\Delta_u = e(u)$.

Theorem (Pickands (1975), Balkema and de Haan (1974))

For a large class of underlying distribution function F, the conditional excess distribution function $F_u(x)$ is well approximated by:

$$F_u(x) \approx W_{\gamma,\beta}(x) \quad u \to \infty.$$

where $W_{\gamma,\beta}(x)$ is the generalized Pareto distribution.

Definition (Pareto distribution)

The generalized *Pareto distribution* (GP) with the parameters $\beta > 0$, γ has the distribution function:

$$W_{\gamma,\beta}(x) = 1 - (1 + \frac{\gamma x}{\beta})^{-\frac{1}{\gamma}} \quad \text{for} \quad \begin{cases} x \geq 0 & \text{if} \quad \gamma > 0 \\ 0 \leq x \leq \frac{-\beta}{\gamma} & \text{if} \quad \gamma < 0, \end{cases}$$

and

$$W_{0,\beta}(x) = 1 - e^{-\frac{1}{\beta}x}, \; x \geq 0.$$

$W_\gamma(x) = W_{\gamma,1}(x)$ are called *generalized standard Pareto distributions* or *standardized GP* distributions.

Submodels of GP distribution

- Exponential (GP0): $W_0(x) = 1 - e^{-x}$, $x \geq 0$
- Pareto (GP1): $W_{1,\beta}(x) = 1 - x^{-\beta}$, $x \geq 1$ and $\beta > 0$
- Beta (GP2): $W_{2,\beta} = 1 - (-x)^{-\beta}, -1 \leq x \leq 0, \beta < 0$

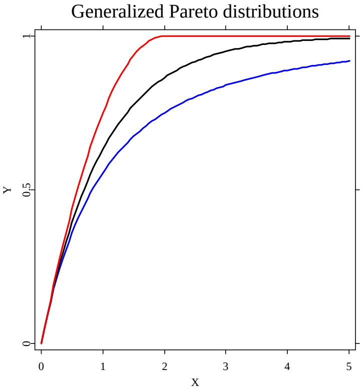
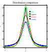


Figure 6: Standard Pareto distributions ($\beta = 1$) with the parameters $\gamma = -0.5$ (Red), 0 (Black) and 0.5 (Blue). SFEgpdist.xpl



Theorem (Mean excess function)

Let X be a positive, unbounded random variable with an absolute continuous distribution function F.

a) *The mean excess function $e(u)$ uniquely determines F:*

$$\bar{F}(x) = \frac{e(0)}{e(x)} \exp\{-\int_0^x \frac{1}{e(u)} du\}, \;\; x > 0.$$

b) *If F belongs to the MDA of the Fréchet distribution $G_{1,\alpha}$, then $e(u)$ is for $u \to \infty$ approximately linear i.e.: $e(u) = \frac{1}{\alpha - 1} \; u\{1 + \mathcal{O}(1)\}$.*

The generalized standard Pareto distribution is the adequate parametric distribution function for exceedances.

Theorem (MDA of GEV distribution)

The distribution F is contained in the MDA of the GEV distribution G_γ with the form parameter $\gamma \geq 0$, exactly when for a measurable function $\beta(u) > 0$ and the GP distribution $W_{\gamma,\beta}$ it holds that:

$$\sup_{x \geq 0} |F_u(x) - W_{\gamma,\beta(u)}(x)| \to 0 \text{ for } u \to \infty.$$

A corresponding result also holds for the case when $\gamma < 0$, in which case the supremum of x must be taken for those $0 < W_{\gamma,\beta(u)}(x) < 1$.

For the generalized Pareto distribution $F = W_{\gamma,\beta}$ it holds for every finite threshold $u > 0$

$$F_u(x) = W_{\gamma,\beta+\gamma u}(x) \quad \text{for} \quad \begin{cases} x \geq 0 & \text{if} \quad \gamma \geq 0 \\ 0 \leq x < -\frac{\beta}{\gamma} - u & \text{if} \quad \gamma < 0, \end{cases}$$

In this case $\beta(u) = \beta + \gamma\, u$.

Estimation in extremes value models

Consider data $x_1, ..., x_m$ generated under a distribution function F^n. Thus each x_i is the maximum of n values that are governed by the distribution function F.

- Gumbel: $G_0(x) = exp\{-e^{-x}\}$. One may use the following two methods to estimate μ and σ of the Gumbel model: $G_{0,\mu,\sigma} = exp\{-e^{-(x-\mu)/\sigma}\}$.
  - MLE: $g_{0,\mu,\sigma} = \frac{1}{\sigma} e^{-(x-\mu)/\sigma} exp(-e^{-(x-\mu)/\sigma})$
  - Moment estimation: estimators μ and σ are deduced from the sample mean $\bar{x}$ and variance s_n. For example, $\sigma_n = \sqrt{6} s_n / \pi$.

- Fréchet model: $G_{1,\alpha}(x) = exp(-x^{-\alpha})$ for $\alpha > 0$ and $x > 0$. MLE can be used. Keep in mind that the left endpoint of $G_{1,\alpha,0,\sigma}$ is always equal to 0.
- Weibull model: $G_{2,\alpha,0,\sigma}$ for $x \leq 0$, $\alpha < 0$ and $\sigma > 0$.

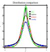
Estimation in Generalized Pareto Models

Let $X_i, i = 1, \ldots, n$ be the original data which are governed by a cdf F.

Notation:

$X_{(1)} \leq \ldots \leq X_{(n)}$ (increasing) order statistics
$X^{(1)} \geq \ldots \geq X^{(n)}$ (decreasing) order statistics
i.e. $X_{(1)} = X^{(n)}, \; X_{(n)} = X^{(1)}$.
We deal with upper extremes which are either

- the exceedances $y_1, ..., y_m$ over a fixed threshold u, or
- the k upper ordered values $y_1, ..., y_m = X^{(1)}, ..., X^{(m)}$.



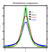
Definition (Empirical Mean Excess Function)

Let $K_n(u) = \{j \leq n;\ X_j > u\}$ be the index set of observations over the threshold value u, set $N(u) = \#K_n(u)$ and define the empirical distribution function as:

$$\hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}(X_j \leq x)$$

straightforwardly, we get $\bar{\hat{F}}_n = 1 - \hat{F}_n$.
The empirical mean excess distribution function is:

$$\begin{aligned} e_n(u) &= \int_u^{\infty} \bar{\hat{F}}_n(y) dy / \bar{\hat{F}}_n(u) \\ &= \frac{1}{N(u)} \sum_{j \in K_n(u)} (X_j - u) = \frac{1}{N(u)} \sum_{j=1}^{n} \max\{(X_j - u), 0\} \end{aligned}$$



For an exploratory data analysis one checks the graphs:

$$\begin{aligned}
&\text{PP-plot} && \left\{F(X^{(k)}),\ \frac{n-k+1}{n+1}\right\}_{k=1}^{n},\\
&\text{QQ-plot} && \left\{X^{(k)},\ F^{-1}\left(\frac{n-k+1}{n+1}\right)\right\}_{k=1}^{n},\\
&\text{mean excess-plot} && \left\{X^{(k)}, e_n(X^{(k)})\right\}_{k=1}^{n}.
\end{aligned}$$

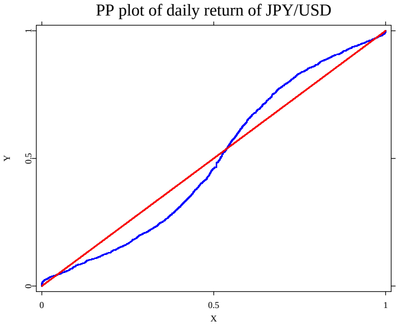


Figure 7: Normal PP plot of daily returns of JPY/USD from 1978-12-01 to 1991-01-31. SFEjpyusd.xpl

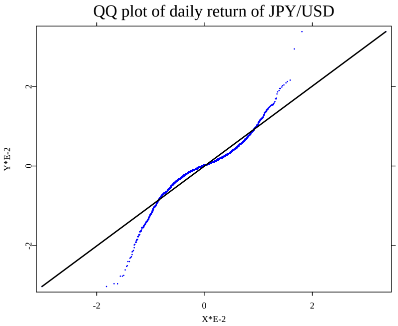


Figure 8: Normal QQ plot of daily returns of JPY/USD from 1978-12-01 to 1991-01-31. SFEjpyusd.xpl

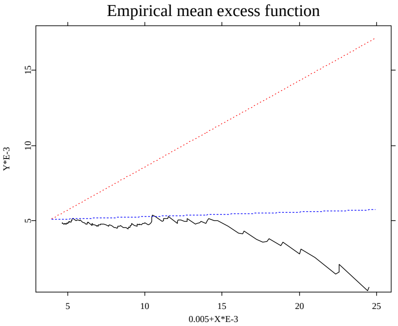


Figure 9: Empirical mean excess plot (solid line), GP mean excess plot for Hill estimator (finely dashed red line) and moment estimator (dashed blue line) of daily returns of JPY/USD from 1978-12-01 to 1991-01-31.
SFEjpyusd.xpl

Nonparametric method

Set $y_1, \ldots, y_m$ are the exceedances over u which are assumed to be iid with cdf F_u.

$$\bar{F}_u(y) = \mathrm{P}(X - u > y \mid X > u) = \bar{F}(y + u)/\bar{F}(u), \quad \text{i.e.}$$
$$\bar{F}(x) = \bar{F}(u) \cdot \bar{F}_u(x - u), \;\; u < x < \infty. \tag{9}$$

For large u and using Theorem (MDA of GEV distribution) we can approximate F_u with $W_{\gamma,\beta}$ by choosing γ and β approximately. $\hat{F}_n(u)$ is replaced by

$$\hat{F}_n(u) = \frac{n - N(u)}{n} = 1 - \frac{N(u)}{n}.$$

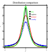
Definition (POT Estimator)

The POT estimator for $\bar{F}(x), x$ large is defined by

$$\bar{F}^{\wedge}(x) = \frac{N(u)}{n}\,\bar{W}_{\hat{\gamma},\hat{\beta}}(x-u) = \frac{N(u)}{n}\left\{1+\frac{\hat{\gamma}(x-u)}{\hat{\beta}}\right\}^{-1/\hat{\gamma}},\ u<x<\infty,$$

where $\hat{\gamma}, \hat{\beta}$ are appropriate estimators for γ, β.

$\hat{\gamma}$ and $\hat{\beta}$ may be computed via the ML method on the basis of the excesses $Y_1, \ldots, Y_{N(u)}$.



MLE of $\hat{\gamma}$ and $\hat{\beta}$

Fix $N(u) = m$ for the moment. $Y_1, \ldots, Y_m$ iid Pareto $W_{\gamma,\beta}, \gamma > 0$, with pdf:

$$p(y) = \frac{1}{\beta}(1 + \frac{\gamma y}{\beta})^{-\frac{1}{\gamma}-1}, \;\; x \geq 0.$$

Log-likelihood:

$$\ell(\gamma, \beta \mid Y_1, \ldots, Y_m) = -m \log \beta - \left(\frac{1}{\gamma} + 1\right) \sum_{j=1}^{m} \log(1 + \frac{\gamma}{\beta} Y_j).$$

Theorem

For all $\gamma > -\frac{1}{2}$, it holds for $m \to \infty$:

$$\sqrt{m}\left(\hat{\gamma} - \gamma,\ \frac{\hat{\beta}}{\beta} - 1\right) \xrightarrow{\mathcal{L}} N_2(0, D^{-1}),$$

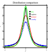
where $D = (1+\gamma)\begin{pmatrix} 1+\gamma & -1 \\ -1 & 2 \end{pmatrix}$, i.e. $(\hat{\gamma}, \hat{\beta})$ are asymptotically normal distributed. The estimators are also asymptotically efficient.

Definition (POT Quantile estimator)

The POT quantile estimator $\hat{x}_q$ for the q-quantile $x_q = F^{-1}(q)$ is the solution of $\bar{F}^{\wedge}(\hat{x}_q) = 1 - q$, i.e.

$$\hat{x}_q = u + \frac{\hat{\beta}}{\hat{\gamma}} \left[\left\{ \frac{n}{N(u)}(1-q) \right\}^{-\hat{\gamma}} - 1 \right].$$

SFEpotquantile.xpl



Comparison to the empirical quantile

Choose u such that $N(u) = m > n(1-q)$, i.e. $u = X^{(m+1)}$.

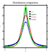
POT quantile estimator:

$$\hat{x}_{q,m} = X^{(m+1)} + \frac{\hat{\beta}_m}{\hat{\gamma}_m}\left[\left\{\frac{n}{m}(1-q)\right\}^{-\hat{\gamma}_m} - 1\right],$$

Empirical quantile: $\hat{x}_q^s = X^{([n(1-q)]+1)}$.
Simulation studies show that

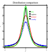
$$m_0 = \operatorname{argmin}_m \mathrm{E}(\hat{x}_{q,m} - x_q)^2$$

is bigger than $[n(1-q)] + 1$. This means that the POT estimator is better than $\hat{x}_q^s$ in MSE terms.



Mean Square Error Dilemma

- u too big: there are not enough exceedances Y and thus the variance is too high.
- u too small: the approximation by Pareto is not good enough and thus a bias occurs.



Theorem

Let Z be a $W_{\gamma,\beta}$ distributed random variable with $0 \leq \gamma < 1$, then the mean excess function of Z is linear:

$$e(u) = \mathrm{E}\{Z - u | Z > u\} = \frac{\beta + \gamma u}{1 + \gamma}, \quad u \geq 0, \quad \textit{for } 0 \leq \gamma < 1.$$

Motivation: Choose u of the POT estimator such that the empirical mean excess function is approximately linear.

Copula

Copula

关联结构

連辞

الارتباط-الصلة

코퓰러

Applications of Copulae for the Calculation of Value-at-Risk

Value-at-Risk (VaR) computation: most VaR methods assume a multivariate normal distribution of the risk factors.

Several pitfalls!

Copulae can be used to describe the dependence between two or more random variables with arbitrary marginal distributions. Backtesting often shows that copule produce more accurate results than "correlation dependence".

Copula, ae [latin]

1.
    a) Band, Leine, Koppel;
    b) Enterhaken
2. Verbindung

关联结构

連辞

الارتباط الصلة

코플러

What is a copula?
A function that links a multidimensional distribution to its one-dimensional margins.
The **joint cumulative distribution functions (cdf)** of d random variables $X_1, \dots, X_d$ with cdf $F_1, \dots, F_d$ is:

$$\begin{aligned} \mathrm{P}(X_1 \le x_1, \dots, X_n \le x_d) &= C\left\{\mathrm{P}(X_1 \le x_1), \dots, \mathrm{P}(X_d \le x_d)\right\} \\ &= C\left\{F_1(x_1), \dots, F_d(x_d)\right\} \end{aligned}$$

Copulae

Definition

A *d-dimensional copula* is a function $C : [0,1]^d \rightarrow [0,1]$:

1. $C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_d) = 0$ (at least one u_i is 0);
2. $u \in [0,1]$, $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ (all coordinates except u_i is 1)
3. For each $u < v \in [0,1]^d$ $(u_i < v_i)$

$$V_C[u,v] = \sum_a sgn(a) C(a) \geq 0$$

where a is taken over all vertices of $[u,v]$. $sgn(a) = 1$ if $a_k = u_k$ for an even number of k's and $sgn(a) = -1$ if $a_k = u_k$ for an odd number of k's (**d-increasing**)

Example

A *2-dimensional copula* is a function $C : [0,1]^2 \to [0,1]$ with the following properties:

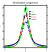
1. For every $u \in [0,1]$, $C(0,u) = C(u,0) = 0$ (**grounded**)
2. For every $u \in [0,1]$, $C(u,1) = u$ and $C(1,u) = u$
3. For every $(u_1, u_2), (v_1, v_2) \in [0,1] \times [0,1]$ with $u_1 \leq v_1$ and $u_2 \leq v_2$: $C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0$ (**2-increasing**)

Copulae

[Sklar's theorem] For a distribution function F with marginals $F_{X_1} \ldots, F_{X_d}$. There exists a copula $C : [0,1]^d \longrightarrow [0,1]$ with

$$F(x_1, \ldots, x_d) = C\{F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\} \tag{10}$$

If $F_{X_1}, \ldots, F_{X_d}$ are cts, then C is unique. If C is a copula and $F_{X_1}, \ldots, F_{X_d}$ are cdfs, then the function F defined in (1) is a joint cdf with marginals $F_{X_1}, \ldots, F_{X_d}$.



Examples of Copulae

Product Copula: independence copula $C = \Pi$ by $\Pi(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i$.
Two random variables X_1 and X_2 are independent if and only if the product of their distributions F_1 and F_2 equals their joint distribution function H, $H(x_1, x_2) = F_1(x_1) \cdot F_2(x_2)$ for all $x_1, x_2 \in \mathbb{R}$.

Let X_1 and X_2 be random variables with continuous distribution functions F_1 and F_2 and joint distribution function H. Then X_1 and X_2 are independent if and only if $C_{X_1 X_2} = \Pi$. According to Sklar's Theorem, there exists a unique copula C with

$$
\begin{aligned}
\mathrm{P}(X_1 \le x_1, X_2 \le x_2) &= H(x_1, x_2) \qquad (11)\\
&= C\left\{F_1(x_1), F_2(x_2)\right\}\\
&= F_1(x_1) \cdot F_2(x_2)
\end{aligned}
$$

Gaussian Copula or normal copula:
d-dimensional with correlation matrix Σ

$$C(\mathbf{u};\Sigma) = \Phi_{\Sigma,d}(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_d))$$

- Φ, univariate standard normal distribution
- $\Phi_{\Sigma,d}$, d-dimensional normal distribution with correlation matrix Σ
- $\mathbf{u} = (u_1,\ldots,u_d)^\top$

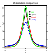
Gaussian Copula or normal copula:

$$C_{\Psi}^{Ga}(u_1,\ldots,u_d) = \Phi_{\Psi}\{\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_d)\}$$

Φ univariate standard normal cdf

Φ_{Ψ} d-dimensional standard normal cdf with correlation matrix Ψ

- Gaussian copula contains *the dependence structure*
- *normal* marginal distributions + Gaussian copula = multivariate normal distributions
- *non-normal* marginal distributions + Gaussian copula = *meta-Gaussian* distributions



Explicit expression for the Gaussian copula

$$
\begin{aligned}
C_{\Psi}^{Ga}(u_1,\ldots,u_d) &= \Phi_{\Psi}\{\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_d)\} \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \ldots \int_{-\infty}^{\Phi^{-1}(u_d)} 2\pi^{-\frac{d}{2}} \mid \Psi \mid^{-\frac{1}{2}} e^{(-\frac{1}{2} r^{\top} \Psi^{-1} r)} dr_1 \ldots dr_d
\end{aligned}
$$

where

$r = (r_1,\ldots,r_d)^{\top}$, $u_j = \Phi(x_j)$

- $C_{\Psi}^{Ga}(u_1,\ldots,u_d)$ allows to generate joint symmetric dependence, but no tail dependence (i.e., there are no joint extreme events)

Example:

$$C_\rho^{\text{Gauss}}(u,v) \stackrel{\text{def}}{=} \int_{-\infty}^{\Phi_1^{-1}(u)} \int_{-\infty}^{\Phi_2^{-1}(v)} f_\rho(x_1,x_2)dx_2dx_1 \ , \tag{12}$$

f_ρ denotes the bivariate normal density function with correlation ρ for $n = 2$.

$$f_\rho(x_1,x_2) = \varphi_\rho(x_1,x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\}.$$

The functions Φ_1, Φ_2 refer to the corresponding marginal cdf.

For $\rho = 0$, the Gaussian copula becomes the product copula.

$$
\begin{aligned}
C_0^{\text{Gauss}}(u, v) &= \int_{-\infty}^{\Phi_1^{-1}(u)} \varphi_1(x_1)dx_1 \int_{-\infty}^{\Phi_2^{-1}(v)} \varphi_2(x_2)dx_2 \\
&= u\,v = \quad \Pi(u, v) \quad \text{if} \quad \rho = 0\,.
\end{aligned}
$$

Replace (u, v) in (12) by $(\Phi(x_1), \Phi(x_1))$, one obtains:

$$
\begin{aligned}
C_\rho^{\text{Gauss}}\{\Phi_1(x_1), \Phi_2(x_2)\} &= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \varphi_\rho(x_1, x_2)dx_2 dx_1 \\
&= \mathrm{P}(X_1 \le x_1, X_2 \le x_2),
\end{aligned}
$$

which is the bivariate cdf of $N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$.

Student's t-copula:
d-dimensional with correlation matrix Σ

$$C(\mathbf{u}; \Sigma, \nu) = T_{\Sigma,\nu}(T_\nu^{-1}(u_1), \ldots, T_\nu^{-1}(u_d))$$

- T_ν, univariate Student's t distribution with ν degrees of freedom and
- $T_{\Sigma,\nu}$, d-dimensional standardized Student's t distribution with ν degrees of freedom and correlation matrix Σ

Frank Copula, $0 < \theta \leq \infty$

$$C_\theta(u_1, \ldots, u_d) = -\frac{1}{\theta} \log \left[1 + \frac{\prod_{j=1}^{d} \{\exp(-\theta u_j) - 1\}}{\{\exp(-\theta) - 1\}^{d-1}} \right]$$

- dependence becomes maximal when $\theta \longrightarrow \infty$
- independence is achieved when $\theta = 0$

Gumbel-Hougaard copula, $1 \le \theta \le \infty$

$$C_\theta(u_1, \dots, u_d) = \exp\left[- \left\{ \sum_{j=1}^{d} (-\log u_j)^\theta \right\}^{\theta^{-1}} \right]$$

- for $\theta > 1$ allows to generate dependence in the upper tail (Schmidt, 2005)
- For $\theta = 1$ reduces to the product copula, i.e. $C_\theta(u_1, \dots, u_d) = \prod_{j=1}^{d} u_j$.
- for $\theta \longrightarrow \infty$, we obtain the Fréchet-Hoeffding upper bound:

$$C_\theta(u_1, \dots, u_d) \stackrel{\theta \to \infty}{\longrightarrow} \min(u_1, , \dots, u_d).$$

Example:
Gumbel-Hougaard Copula:

$$C_\theta(u,v) \stackrel{\text{def}}{=} \exp\left[-\left\{(-\ln u)^\theta + (-\ln v)^\theta\right\}^{1/\theta}\right]. \qquad (13)$$

The parameter θ may take all values in the interval $[1,\infty)$.*For* $\theta = 1$, *Gumbel-Hougaard Copula reduces to the product copula, i.e.* $C_1(u,v) = \Pi(u,v) = u\,v$. *For* $\theta \to \infty$, *Gumbel-Hougaard copula changes to* $C_\theta(u,v) \stackrel{\theta\to\infty}{\longrightarrow} \min(u,v) \stackrel{\text{def}}{=} M(u,v)$, *where* M *is also a copula.* This copula family is suited for bivariate extreme value distribution.

Ali-Mikhail-Haq copula, $-1 \leq \theta < 1$

$$C_\theta(u_1, \ldots, u_d) = \frac{\prod\limits_{j=1}^{d} u_j}{1 - \theta \left\{ \prod\limits_{j=1}^{d} (1 - u_j) \right\}}$$

- independence is achieved when $\theta = 0$
- the Fréchet-Hoeffding bounds are not achieved

Emil Julius Gumbel

born on July 18, 1891 in München, Germany
died on September 10, 1966 in New York, USA

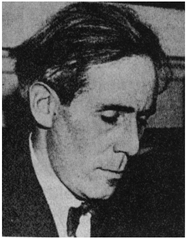


Figure 10:

Born and trained as a statistician in Germany, he was forced to move to France and then the U.S. because of his pacifist and socialist views. He was a pioneer in the application of extreme value theory, particularly to climate and hydrology. The Gumbel distribution is named after him.

Maurice Fréchet [1878-1973]
born on September 2, 1878 in Maligny, Yonne, Bourgogne, France
died on June 4, 1973 in Paris, France

Figure 11:

French mathematician who made major contributions to pure mathematics as well as probability and statistics. He also collected empirical examples of heavy-tailed distributions. The Fréchet type of extreme value distribution is named after him (this distribution has a heavy tail).

Wassily Hoeffding

born on June 12, 1914 in Mustamäki, Finland (U.S.S.R. since 1940)
died on February 28, 1991 in Chapel Hill, USA

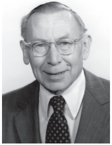


Figure 12:

He spend his childhood in Tsarskoye Selo, Ukraine and in Denmark. In 1924 the family settled in Berlin. He entered Berlin University to study mathematics in 1934. In 1946 he moved to the USA where he was offered a position of a research associate at the Department of Mathematical Statistics at the University of North Carolina at Chapel Hill in 1947. He remained in Chapel Hill for the rest of his life.

Wassily Hoeffding

Tsarskoe Selo was, of course, Wassily's hometown.

Куда бы нас ни бросила судьбина,
И счастие куда б ни повело,
Всё те же мы: нам целый мир чужбина;
Отечество нам Царское Село.

А.С.Пушкин

For the whole world is a strange country,
Our motherland is Tsarskoe Selo.

A. Pushkin

Figure 13:

$M(u, v) = min(u, v)$ **is a copula.**

1. $M(0, v) = 0 = M(u, 0)\ \forall u \in [0, 1]$, thus it is grounded.
2. $M(u, 1) = u$ and $M(1, v) = v$
3. $u_1 \leq v_1$, $u_2 \leq v_2$:
   - $v_1 \leq u_2$: $M(v_1, v_2) - M(v_1, u_2) - M(u_1, v_2) + M(u_1, u_2)$
     $= v_1 - v_1 - u_1 + u_1 = 0$
   - $u_1 \leq u_2 \leq v_1 \leq v_2$: $v_1 - u_2 - u_1 + u_1 \geq 0$

   yield 2-increasing property.

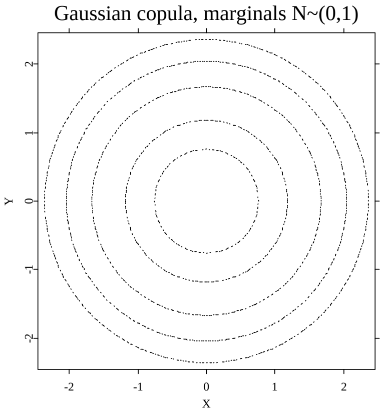


Figure 14: Contour plots of pdf from $F(x_1, x_2) = C(\Phi(x_1), \Phi(x_2))$ with Gaussian copula.

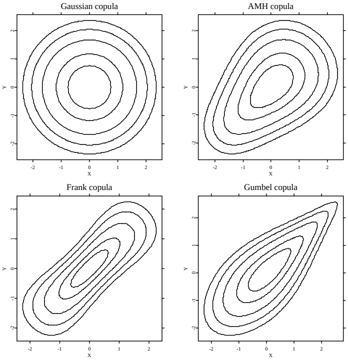


Figure 15: Contour plots of pdf from $F(x_1, x_2) = C(\Phi(x_1), \Phi(x_2))$ with Gaussian, AMH, Frank and Gumbel-Hougaard copulae.

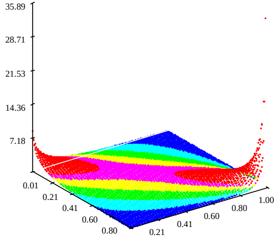


Figure 16: Density from Gumbel-Hougaard copula, $\theta = 2$.

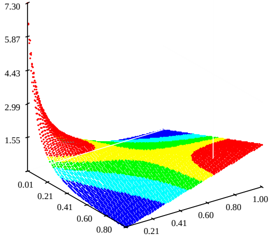
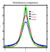


Figure 17: Density from AMH copula, $\theta = 0.9$.



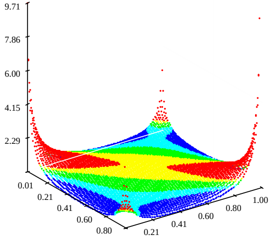


Figure 18: Density from t-copula, $\rho = 0.2$, $\nu = 3$.

Important Properties of Copulae

Important Properties of Copulae

- **Fréchet-Hoeffding upper bound M:** for any given copula C one has $C(u,v) \le M(u,v) = \min(u,v)$.
- **Fréchet-Hoeffding lower bound W:** Two-dimensional function $W(u,v) \overset{\text{def}}{=} \max(u+v-1,0) \le C(u,v)$.

Theorem

Let C be a copula. Then for every $u_1, u_2, v_1, v_2 \in [0,1]$:

$$|C(u_2,v_2) - C(u_1,v_1)| \le |u_2 - u_1| + |v_2 - v_1|\,. \tag{14}$$

Theorem

Let C be a copula. For every $u \in [0,1]$, the partial derivative $\partial C/\partial v$ exists for almost every $v \in [0,1]$. For such u and v one has

$$0 \le \frac{\partial}{\partial v} C(u,v) \le 1\,. \tag{15}$$

The analogous statement is true for the partial derivative $\partial C/\partial u$.

Example: partial derivative of the Gumbel-Hougaard copula

$$C_{\theta,u}(v) = \frac{\partial}{\partial u} C_\theta(u,v) = \exp\left\{ - \left[(-\ln u)^\theta + (-\ln v)^\theta \right]^{1/\theta} \right\} \times$$
$$\left[(-\ln u)^\theta + (-\ln v)^\theta \right]^{-\frac{\theta-1}{\theta}} \;\frac{(-\ln u)^{\theta-1}}{u}. \tag{16}$$

$C_{\theta,u}$ is a strictly increasing function of v for $u \in (0,1)$ and for all $\theta \in \mathbb{R}$ where $\theta > 1$, . Therefore the inverse function $C^{-1}_{\theta,u}$ is well defined. Numerical algorithm has to be used for the calculation.

Theorem

Let X_1 and X_2 be random variables with continuous distribution functions and with copula $C_{X_1X_2}$. If α_1 and α_2 are strictly increasing functions on Range X_1 and Range X_2, then $C_{\alpha_1(X_1)\,\alpha_2(X_2)} = C_{X_1X_2}$. In other words, $C_{X_1X_2}$ is invariant under strictly increasing transformations of X_1 and X_2.

Value-at-Risk with Copulae

For a sample $\{X_t\}_{t=1}^{T}$

1. specification of marginal distributions $F_{X_j}(x_j; \delta_j)$
2. specification of copula $C(u_1, \ldots, u_d; \theta)$
3. fit of the copula C
4. generation of Monte Carlo data
   $X_{T+1} \sim C\{F_1(x_1), \ldots, F_d(x_d); \hat{\theta}\}$
5. generation of a sample of portfolio losses $L_{T+1}(X_{T+1})$
6. estimation of $\widehat{VaR}(\alpha)$, the empirical quantile at level α from $L_{T+1}(X)$.

For copulae $C(\cdot, \theta)$, $\theta \in \Theta$, the density of X is given by:

$$f(x_1, \dots, x_d; \delta_1, \dots, \delta_d, \theta) =$$

$$= c\{F_{X_1}(x_1; \delta_1), \dots, F_{X_d}(x_d; \delta_d); \theta\} \prod_{j=1}^{d} f_j(x_j; \delta_j)$$

where

$$c(u_1, \dots, u_d) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d}$$

Inference for Margins

In the IFM (*inference for margins*) method, the log-likelihood function for each of the marginal distributions

$$\ell_j(\delta_j) = \sum_{t=1}^{T} \ln f_i(x_{j,t}; \delta_j), j = 1, \dots, d$$

is maximized to obtain estimates $(\hat{\delta}_1, \dots, \hat{\delta}_d)^\top$.

The function

$$\ell(\theta, \hat{\delta}_1, \ldots, \hat{\delta}_d) = \sum_{t=1}^{T} [\ln c\{F_{X_1}(x_{1,t}; \hat{\delta}_1), \ldots, F_{X_d}(x_{d,t}; \hat{\delta}_d); \theta\}]$$

is then maximized over θ to get the dependence parameter estimate $\hat{\theta}$. The estimates $\hat{\theta}_{IFM} = (\hat{\delta}_1, \ldots, \hat{\delta}_d, \hat{\theta})^\top$ solve

$$(\partial \ell_1 / \partial \delta_1, \ldots, \partial \ell_d / \partial \delta_d, \partial \ell / \partial \theta) = 0$$

Static Approach

- DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994
- log returns are assumed to be $X_{j,t} \sim N(0, \sigma_j)$, $j = 1, 2$
- σ_j estimated from the data
- $T = 3719$
- copulae belong to the bivariate one-parametric Gumbel family

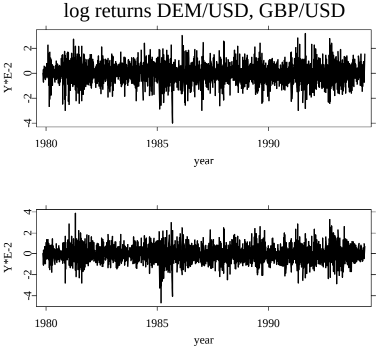


Figure 19: Log returns from DEM/USD (X_1) and GBP/USD (X_2).

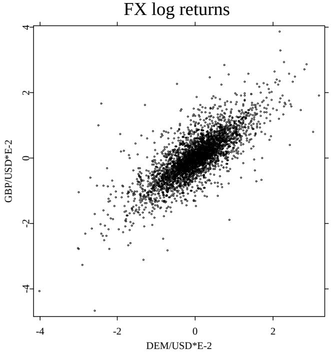


Figure 20: Scatterplot from log returns DEM/USD (X_1) and GBP/USD (X_2).

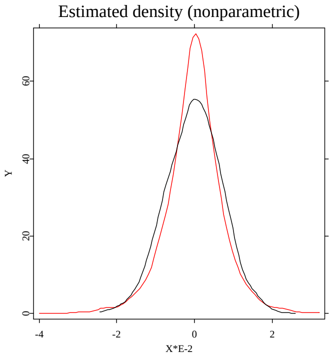
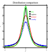


Figure 21: Kernel density estimator of the log returns from DEM/USD (red) and of the normal density (black). Quartic kernel, $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.



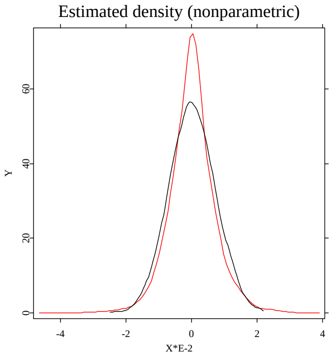


Figure 22: Kernel density estimator of the log returns from GBP/USD (red) and of the normal density (black). Quartic kernel, $\hat{h} = 2.78\hat{\sigma}n^{-0.2}$.

Dependence

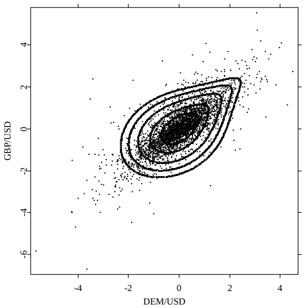


Figure 23: Standardised log returns DEM/USD and GBP/USD, fitted copula ($\hat{\theta} = 1.4461$) for $T = 3719$.

Value-at-Risk

	level $\alpha(\times 10^2)$			
	5	1	0.5	0.1
$\widehat{VaR}(\alpha)$	-0.02436	-0.034115	-0.037921	-0.042611

Table 1: Estimated Value-at-Risk at 4 different levels, FX portfolio, $w = (1,1)^\top$.

Moving window

- DEM/USD and GBP/USD from 01.12.1979 to 01.04.1994
- sample size $S = 3719$, time window $T = 250$, for $s = T + 1, \ldots, S$
- using $\{X_t\}_{t=s-T}^{s}$
- log returns are assumed to be $X_{j,t} \sim N(0, \sigma_j)$, $j = 1, 2$
- σ_j estimated from the data
- copulae belong to the bivariate one-parametric Gumbel family

Parameter $\hat{\sigma}_1$ from marginal distribution

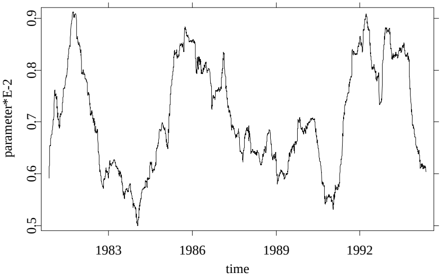


Figure 24: Estimated parameter from Normal marginal distribution $\hat{\sigma}_1$ for log returns from DEM/USD, $T = 250$.

Parameter $\hat{\sigma}_2$ from marginal distribution

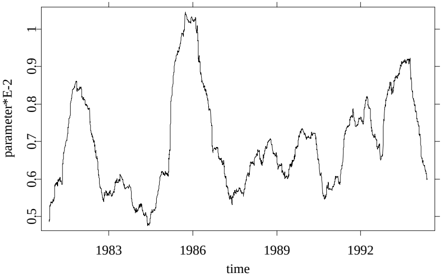


Figure 25: Estimated parameter from Normal marginal distribution $\hat{\sigma}_2$ for log returns from GBP/USD, $T = 250$.

Copula parameter $\hat{\theta}$

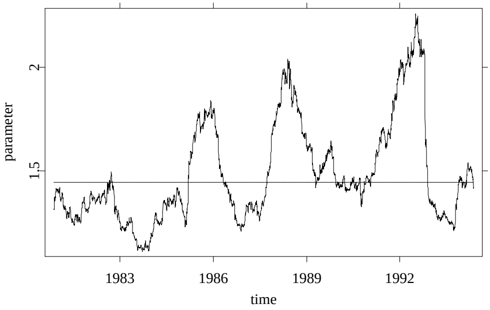


Figure 26: Gumbel dependence parameter $\hat{\theta}$ between DEM/USD and GBP/USD (standardised log returns). Estimated with Normal marginal distributions using IFM method, $T = 250$ (constant value for $T = 3719$).

	min	max	mean	median	std error
$\hat{\sigma}_1.10^3$	4.99	9.12	7.09	6.91	1.02
$\hat{\sigma}_2.10^3$	4.74	10.46	6.95	6.69	1.31
$\hat{\theta}$	1.11	2.25	1.48	1.42	0.24

Table 2: Descriptive statistics for estimated parameters $\hat{\sigma}_1$, $\hat{\sigma}_2$ and $\hat{\theta}$.

Minimal and maximal dependence

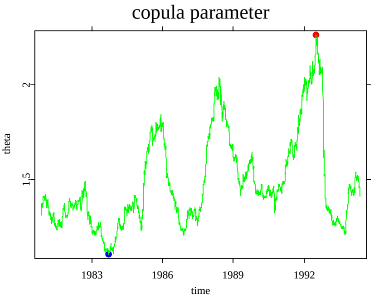


Figure 27: Minimal (blue), maximal (red) dependence parameter between standardised log returns DEM/USD and GBP/USD.

Minimal dependence

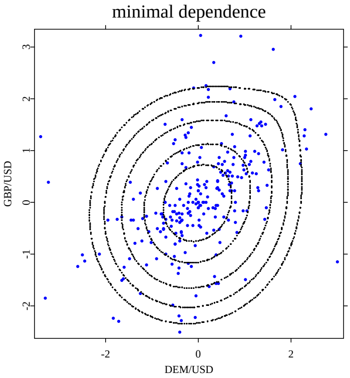


Figure 28: Standardised log returns DEM/USD and GBP/USD at minimal dependence (blue), fitted copula ($\hat{\theta} = 1.11$).

Maximal dependence

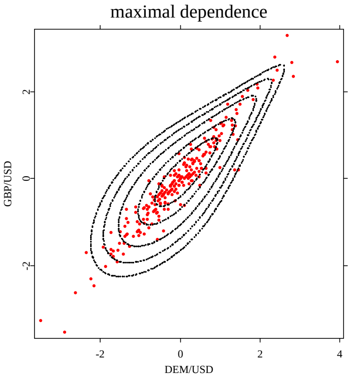


Figure 29: Standardised log returns DEM/USD and GBP/USD at maximal dependence (red), fitted copula ($\hat{\theta} = 2.25$).

MC sample, minimal dependence

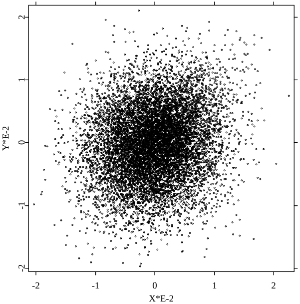


Figure 30: Monte Carlo sample of random variables $X \sim C\{\Phi_1(x_1), \Phi_2(x_2); \hat{\theta}\}$, minimal dependence ($\hat{\theta} = 1.11$).

Transformed MC sample, minimal dependence

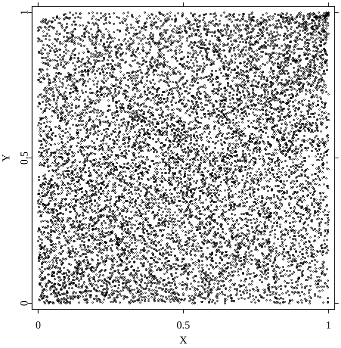


Figure 31: Monte Carlo sample of random variables transformed on the unit square, minimal dependence ($\hat{\theta} = 1.11$).

MC sample, maximal dependence

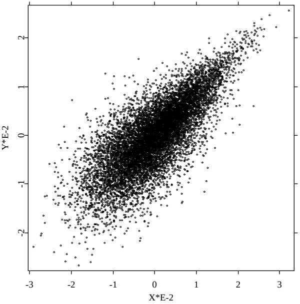


Figure 32: Monte Carlo sample of random variables $X \sim C\{\Phi_1(x_1), \Phi_2(x_2); \hat{\theta}\}$, maximal dependence ($\hat{\theta} = 2.25$).

Transformed MC sample, maximal dependence

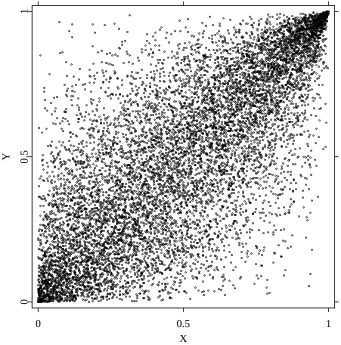


Figure 33: Monte Carlo sample of random variables transformed on the unit square, maximal dependence ($\hat{\theta} = 2.25$).

Backtesting

Evaluation:

- different portfolio compositions are used
- the VaR $\alpha = 0.05$, $\alpha = 0.01$, $\alpha = 0.005$ and $\alpha = 0.001$ is calculated
- *exceedance* for each P&L value smaller than VaR

Value-at-Risk

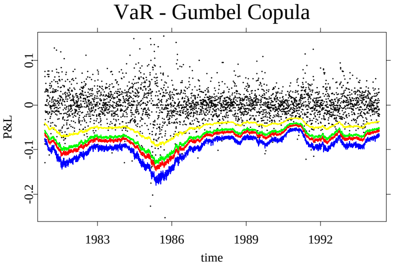
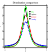


Figure 34: Value-at-Risk at levels $\alpha_1 = 0.05$ (yellow), $\alpha_2 = 0.01$ (green), $\alpha_3 = 0.005$ (red), and $\alpha_4 = 0.001$ (blue), P&L (black), $w = (2,1)^\top$, estimated at each time from a Monte Carlo sample of 10.000 P&L values.



Value-at-Risk (0.05) and exceedances

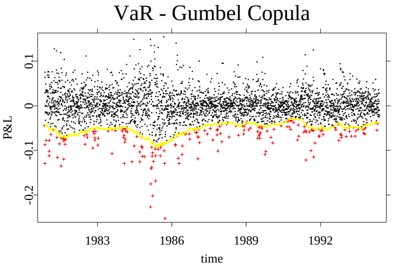


Figure 35: Value-at-Risk (yellow) at level $\alpha = 0.05$, P&L (black) and exceedances (red), $\hat{\alpha} = 0.0573$, $w = (2, 1)^\top$. P&L samples generated with Gumbel-Hougaard copula.

Value-at-Risk (0.001) and exceedances

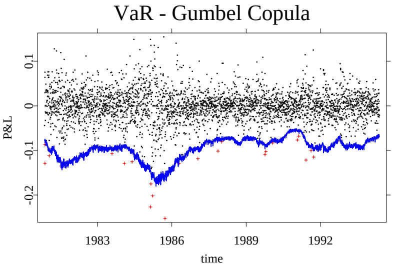


Figure 36: Value-at-Risk (blue) at level $\alpha = 0.001$, P&L (black) and exceedances (red), $\hat{\alpha} = 0.0069$, $w = (2,1)^\top$. P&L samples generated with Gumbel-Hougaard copula.

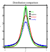
Portfolio $w^\top$	level $\alpha(\times 10^2)$ 5	1	0.5	0.1
	empirical level $\hat{\alpha}(\times 10^2)$			
$(1,1)$	6.05	2.45	1.75	0.83
$(1,2)$	6.34	2.74	1.75	1.00
$(2,1)$	5.73	2.24	1.58	0.69
$(2,3)$	6.22	2.56	1.75	0.92
$(3,2)$	5.99	2.30	1.55	0.74
$(-1,2)$	1.64	0.37	0.20	0.11
$(1,-2)$	2.01	0.51	0.43	0.11
$(-2,1)$	4.44	1.49	0.95	0.40
$(2,-1)$	4.09	1.35	1.09	0.49

Table 3: Gumbel-Hougaard copula, empirical levels $\hat{\alpha}$ for different FX portfolios.

Negative Log-returns

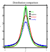
Portfolio $w^\top$	level $\alpha(\times 10^2)$			
	5	1	0.5	0.1
	empirical level $\hat{\alpha}(\times 10^2)$			
$(1,1)$	5.25	1.82	1.15	0.63
$(1,2)$	5.39	1.64	1.24	0.60
$(2,1)$	5.27	1.79	1.27	0.66
$(2,3)$	5.30	1.70	1.21	0.66
$(3,2)$	5.27	1.78	1.26	0.66
$(-1,2)$	1.41	0.29	0.23	0.05
$(1,-2)$	2.74	0.98	0.61	0.28
$(-2,1)$	4.32	1.15	0.79	0.26
$(2,-1)$	4.49	1.67	1.24	0.69

Table 4: Gumbel-Hougaard copula on negative log-returns, empirical levels $\hat{\alpha}$ for different FX portfolios.



DAX-Dow Jones portfolio

- DAX and Dow Jones from 02.01.1997 to 30.12.2004
- sample size $S = 2022$, time window $T = 250$, for $s = T + 1, \ldots, S$
- using $\{X_t\}_{t=s-T}^{s}$
- log returns are assumed to be $X_{j,t} \sim N(0, \sigma_j)$, $j = 1, 2$
- σ_j estimated from the data
- copulae belong to the bivariate one-parametric Gumbel-Hougaard family



	level $\alpha(\times 10^2)$			
	5	1	0.5	0.1
Portfolio $w^\top$	empirical level $\hat{\alpha}(\times 10^2)$			
$(1,1)$	4.28	1.29	0.84	0.45
$(1,2)$	3.89	1.29	0.79	0.50
$(2,1)$	4.62	1.52	0.90	0.56
$(2,3)$	4.06	1.18	0.73	0.50
$(3,2)$	4.57	1.46	0.90	0.62
$(-1,2)$	5.07	1.52	0.84	0.39
$(1,-2)$	4.79	1.58	1.24	0.45
$(-2,1)$	4.96	1.46	0.95	0.39
$(2,-1)$	4.96	1.74	1.12	0.62

Table 5: Gumbel-Hougaard copula, empirical levels $\hat{\alpha}$ for different DAX Dow Jones portfolios.

DAX - Dow Jones: Value-at-Risk (0.05) and exceedances

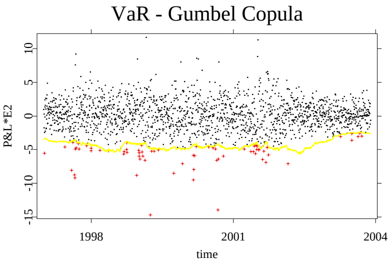


Figure 37: Value-at-Risk (yellow) at level $\alpha = 0.05$, P&L (black) and exceedances (red), $\hat{\alpha} = 038939$, $w = (1, 2)^\top$. P&L samples generated with Gumbel-Hougaard copula.

Adaptive Copulae

In the local homogeneity modelling the copula parameter is a piecewise constant function θ_t

- search for largest interval $I = [n-m, n[$ that does not contain a change point,

$$\theta_t = \theta_I, t \in I$$

- within I, θ_n can be estimated through

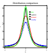
$$\tilde{\theta}_I = \arg\max_{\theta} L_I(\theta)$$

where $L_I(\theta) = \sum_{i \in I} \ell(x_i; \theta)$.

Determining I

The homogeneity interval I can be determined as follows

- select a set $\mathcal{I}$ of candidate intervals
- take the smallest $I \in \mathcal{I}$
- test homogeneity in I against change-point alternative
- if rejected at point $\nu \in I$, $\hat{I} = [\nu, n[$
- if not rejected, choose larger I



Change-point Test

- $\mathcal{T}(I)$ a family of internal points of I
- each $\tau \in \mathcal{T}(I)$ splits the interval I into sub-intervals $J = [n-\tau, n[$ and $J^c = [n-m, n-\tau[$
- likelihood ratio test statistic for change-point at τ

$$T_{I,\tau} = L_J(\tilde{\theta}_J) + L_{J^c}(\tilde{\theta}_{J^c}) - L_I(\tilde{\theta}_I)$$

- change-point test

$$T_{I,\nu} = \max_{\tau} T_{I,\tau}$$

If $T_{I,\nu} \geq \lambda_I$, reject homogeneity and

- ν is change-point time
- $\hat{I} = [\nu, n[$ is the homogeneity interval
- $\tilde{\theta} = \arg\max_{\theta} L_{\hat{I}}(\theta)$ the estimated copula parameter.

Critical Value λ_I

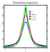
Adaptive procedure, type I error (α): multiple testing problem

- for each I, define β_I and α_I such that

$$\sum_{I \in \mathcal{I}} \beta_I = \alpha$$

$$\alpha_I = \sum_{I' \in \mathcal{I}(I)} \beta_{I'}$$

where $\mathcal{I}(I) = \{I' : I' \in \mathcal{I}, I' \subseteq I\}$



- within I: change-point test at level α_I
- $n = 5000$ Monte Carlo simulations of T_I
- λ_I is $(1 - \alpha_I)$-quantile of computed test statistics T_I

Monte Carlo Simulation

Gumbel-Hougaard copulae simulated with parameter:

$$\theta_{1,t} = \begin{cases} 1 & : \quad 1 \leq t \leq 60 \\ 5 & : \quad 61 \leq t \leq 120 \\ 1 & : \quad 121 \leq t \leq 180 \end{cases}$$

and

$$\theta_{2,t} = \begin{cases} 1.5 & : \quad 1 \leq t \leq 260 \\ 6 & : \quad 261 \leq t \leq 320 \\ 3 & : \quad 321 \leq t \leq 380 \\ 1 & : \quad 381 \leq t \leq 440 \end{cases}$$

- set of candidate intervals

$$\mathcal{I} = \{I_k : I_k = [t - m_k, t[\}$$

$$m_k = [m_0 c^k], k = 0, 1, 2$$

- $[x]$ is the integer part of x
- defining β_{I_k} as

$$\beta_{I_k} = \frac{\alpha}{m_k} \left(\sum_{l=1}^{\infty} m_l^{-1} \right)^{-1} \approx \frac{\alpha(1 - c^{-1})}{c^k}$$

- defining α_{I_k} as

$$\alpha_{I_k} \approx \left(1 - c^{-(k+1)}\right)$$

- critical values λ_{I_k} are obtained through Monte Carlo simulation.
- values set to $m_0 = 30$, $c = 2$ and $\alpha = 0.05$

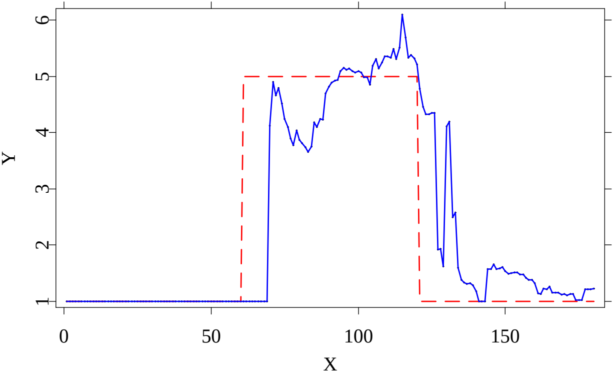


Figure 38: Real parameter $\theta_{1,t}$ (red) and estimated (blue).

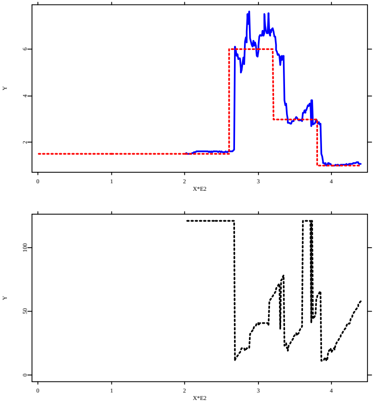


Figure 39: Real parameter $\theta_{2,t}$ (red), estimated (blue) and interval $\hat{I}$ (black).

Joint Extreme Value

Gemeinsamer Extremwert

联合极值

極值

القيمة الحدية المشتركة

극단값

Measures of dependence

- Pearson's correlation coefficient ρ
- Kendall's τ
- Spearman's rank correlation coefficient ρ_S

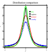
Correlation $\rho(X, Y) = \frac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$ measures linear dependence

Pearson's ρ

Makes sense only for finite variance (for extreme value distributions e.g. Fréchet it cannot be applied)
Correlation is not universal w.r.t measure transformation:
$\rho(X, Y) \neq \rho(\log X, \log Y)$.

$$\rho(X, Y) = \{Var(X)Var(Y)\}^{-1/2} \int_0^1 \int_0^1 \{C(u, v)\} dF^{-1}(u) G^{-1}(v)$$

ρ depends on scale of X and Y.



Tail Dependence

Risk behavior is determined by tails large losses that can occur jointly.
Pearson's correlation can not capture joint large loss events.
Tail dependence describes the limiting proportion that one margin exceeds a certain threshold given that the other margin has already exceeded that threshold.

What is tail dependence?

For $X = (X_1, X_2)^\top \in \mathbb{R}$ define
upper tail dependence as:

$$\lambda_U \stackrel{\text{def}}{=} \lim_{v \uparrow 1} \mathrm{P}\left\{X_1 > F_1^{-1}(v) \mid X_2 > F_2^{-1}(v)\right\} > 0, \qquad (17)$$

F_i^{-1} are the generated inverse cdfs:

$$x = F^{-1}(u) = \sup\{x : F(x) \leq u\}$$

$\lambda_U = 0$, upper tail independent.

Similarly, define the lower tail dependence coefficient (TDC):

$$\lambda_L \stackrel{\text{def}}{=} \lim_{v \downarrow 0} \mathrm{P}\left\{X_1 \leq F_1^{-1}(v) \mid X_2 \leq F_2^{-1}(v)\right\}. \tag{18}$$

Example

$X \sim N_2(0, \Sigma)$ or $X \sim t(p)$

$$\lambda_U = \lim_{v \uparrow 1} \lambda_U(v) \stackrel{\text{def}}{=} \lim_{v \uparrow 1} 2 \cdot \mathrm{P}\left\{X_1 > F_1^{-1}(v) \mid X_2 = F_2^{-1}(v)\right\}. \tag{19}$$

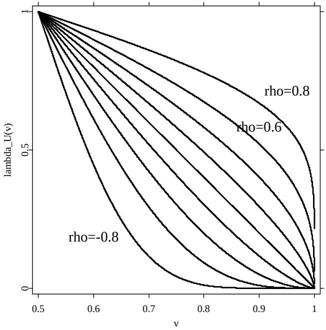


Figure 40: The function $\lambda_U(v) = 2 \cdot \mathrm{P}\{X_1 > F_1^{-1}(v) \mid X_2 = F_2^{-1}(v)\}$ for a bivariate normal distribution with correlation coefficients $\rho = -0.8, -0.6, \ldots, 0.6, 0.8$. Note that $\lambda_U = 0$ for all $\rho \in (-1, 1)$. STFtail01.xpl

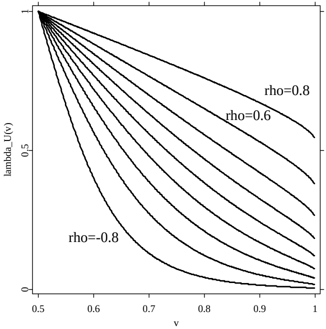


Figure 41: The function $\lambda_U(v) = 2 \cdot \mathrm{P}\{X_1 > F_1^{-1}(v) \mid X_2 = F_2^{-1}(v)\}$ for a bivariate t-distribution (3 df) with correlation coefficients $\rho = -0.8, -0.6, \ldots, 0.6, 0.8$. STFtail02.xpl

The TDC can be expressed in forms of copulae.

$$F(x_1, x_2, ..., x_d) = C\{F_1(x_1), ..., F_d(x_d)\}$$

If X is continuous:

$$\begin{aligned} \lambda_U &= \lim_{v \uparrow 1} \frac{1 - 2v + C(v, v)}{1 - v}, \\ \lambda_L &= \lim_{v \downarrow 0} \frac{C(v, v)}{v} \end{aligned} \tag{20}$$

TDCs for Archimedean copulae

Archimedean copula:

$$C(u,v) = \psi^{[-1]}\{\psi(u) + \psi(v)\}$$

for some cts, decreasing and convex ψ, $\psi(1) = 0$.

$$\psi^{[-1]}(t) = \begin{cases} \psi^{-1}(t), & 0 \le t \le \psi(0), \\ 0, & \psi(0) < t \le \infty. \end{cases}$$

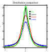
For $\psi(0) = \infty$: $\psi^{[-1]} = \psi^{-1}$.

Table 6: Various selected Archimedean copulae. The numbers in the first column correspond to the numbers of Table 4.1 in Nelsen (1999), p. 94.

Number & Type		$C(u,v)$	Parameters
(1)	Clayton	$\max\left\{(u^{-\theta}+v^{-\theta}-1)^{-1/\theta},0\right\}$	$\theta \in [-1,\infty)\backslash\{0\}$
(2)		$\max\left[1-\left\{(1-u)^{\theta}+(1-v)^{\theta}\right\}^{1/\theta},0\right]$	$\theta \in [1,\infty)$
(3)	Ali-Mikhail-Haq	$\frac{uv}{1-\theta(1-u)(1-v)}$	$\theta \in [-1,1)$
(4)	Gumbel-Hougaard	$\exp\left[-\{(-\log u)^{\theta}+(-\log v)^{\theta}\}^{1/\theta}\right]$	$\theta \in [1,\infty)$
(12)		$\left[1+\left\{(u^{-1}-1)^{\theta}+(v^{-1}-1)^{\theta}\right\}^{1/\theta}\right]^{-1}$	$\theta \in [1,\infty)$
(14)		$\left[1+\left\{(u^{-1/\theta}-1)^{\theta}+(v^{-1/\theta}-1)^{\theta}\right\}^{1/\theta}\right]^{-\theta}$	$\theta \in [1,\infty)$
(19)		$\theta/\log\left(e^{\theta/u}+e^{\theta/v}-e^{\theta}\right)$	$\theta \in (0,\infty)$

Table 7: Tail-dependence coefficients (TDCs) and generators ψ_θ for various selected Archimedean copulae. The numbers in the first column correspond to the numbers of Table 4.1 in Nelsen (1999), p. 94.

Number & Type	$\psi_\theta(t)$	Parameter θ	Upper-TDC	Lower-TDC
(1) Pareto	$t^{-\theta} - 1$	$[-1, \infty)\backslash\{0\}$	0 for $\theta > 0$	$2^{-1/\theta}$ for $\theta > 0$
(2)	$(1-t)^\theta$	$[1, \infty)$	$2 - 2^{1/\theta}$	0
(3) Ali-Mikhail-Haq	$\log \frac{1-\theta(1-t)}{t}$	$[-1, 1)$	0	0
(4) Gumbel-Hougaard	$(-\log t)^\theta$	$[1, \infty)$	$2 - 2^{1/\theta}$	0
(12)	$\left(\frac{1}{t} - 1\right)^\theta$	$[1, \infty)$	$2 - 2^{1/\theta}$	$2^{-1/\theta}$
(14)	$\left(t^{-1/\theta} - 1\right)^\theta$	$[1, \infty)$	$2 - 2^{1/\theta}$	$\frac{1}{2}$
(19)	$e^{\theta/t} - e^\theta$	$(0, \infty)$	0	1



Estimation of the TDC:
$\{X_j\}_{j=1}^n \in \mathbb{R}^2$ i.i.d. the empirical copula is

$$C_n(u,v) = F_n(F_{1n}^{-1}(u), F_{2n}^{-1}(v)),$$

F_{in} empirical cdfs of X_{ij}, $j = 1, ..., n$.

$$\begin{aligned}\hat{\lambda}_{U,n}^{(1)} &= \frac{n}{k} C_n\Big(\Big(1-\frac{k}{n},1\Big] \times \Big(1-\frac{k}{n},1\Big]\Big) \\ &= \frac{1}{k}\sum_{j=1}^{n} I(R_{n1}^{(j)} > n-k, R_{n2}^{(j)} > n-k)\end{aligned}$$

Here $R_{n1}^{(j)}$ and $R_{n2}^{(j)}$ is the rank of $X_1^{(j)}$ and $X_2^{(j)}$ respectively.

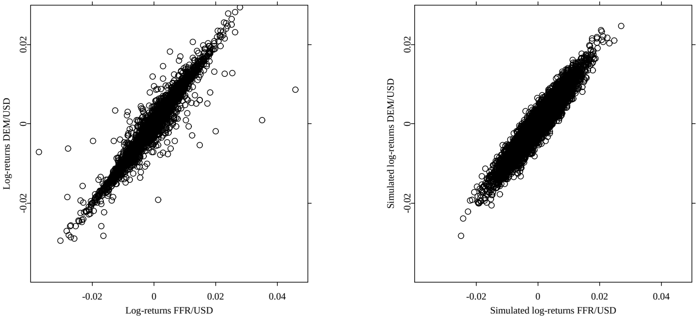
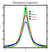


Figure 42: Scatter plot of foreign exchange data (left panel) and simulated normal pseudo-random variables (right panel) of FFR/USD versus DEM/USD negative daily exchange rate log-returns (5189 data points).
STFtail08.xpl



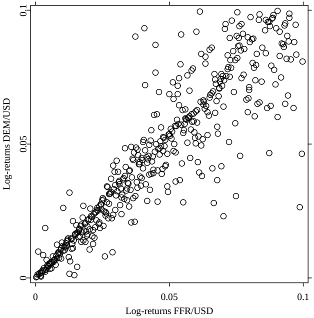


Figure 43: Lower left corner of the empirical copula density plots of real data (left panel) and simulated normal pseudo-random variables (right panel) of FFR/USD versus DEM/USD negative daily exchange rate log-returns (5189 data points). STFtail09.xpl

Lower TDC estimate:

$$\hat{\lambda}^{(1)}_{L,n} = \frac{n}{k} C_n\Big(\frac{k}{n}, \frac{k}{n}\Big) = \frac{1}{k}\sum_{j=1}^{n} I(R^{(j)}_{n1} \le k, R^{(j)}_{n2} \le k), \qquad (21)$$

where $k = k(n) \to \infty$ and $k/n \to 0$ as $m \to \infty$,
From EVT:

$$\begin{aligned} \hat{\lambda}^{(2)}_{U,n} &= 2 - \frac{n}{k}\Big\{1 - C_n\Big(1 - \frac{k}{n}, 1 - \frac{k}{n}\Big)\Big\} \\ &= 2 - \frac{1}{k}\sum_{j=1}^{n} I(R^{(j)}_{n1} > n - k \text{ or } R^{(j)}_{n2} > n - k), \qquad (22) \end{aligned}$$

obtains the usual nonparametric bias-variance problem.

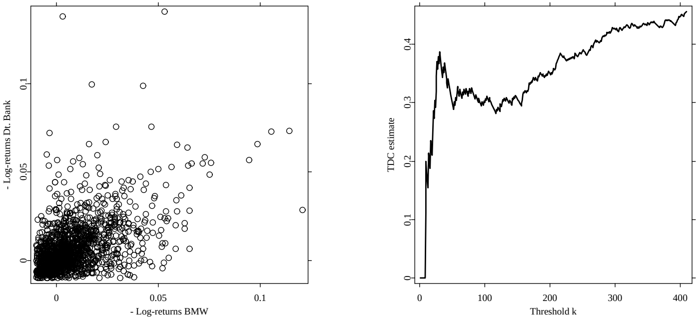


Figure 44: Scatter plot of BMW versus Deutsche Bank negative daily stock log-returns (2347 data points) and the corresponding TDC estimate $\hat{\lambda}_U^{(1)}$ for various thresholds k. Chosen $k \approx 90$, TDC $\hat{\lambda}_U^{(1)} = 0.31$.
STFtail06.xpl

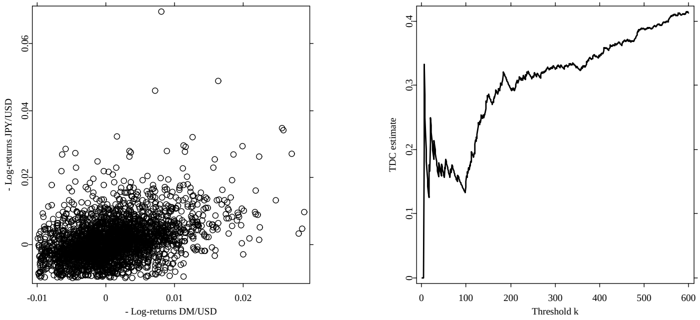


Figure 45: Scatter plot of DEM/USD versus JPY/USD negative daily exchange rate log-returns (3126 data points) and the corresponding TDC estimate $\hat{\lambda}_U^{(1)}$ for various thresholds k. Chosen $k \approx 60$, TDC $\hat{\lambda}_U^{(1)} = 0.17$.
STFtail07.xpl

VaR simulation study

Data (daily log returns)

$D1$:	BMW-Deutsche Bank	(1992-2001)
$D2$:	FX DEM/USD and JPY/USD	(1989-2001)
$D3$:	FX FFR/USD and DEM/USD	(1984-2002)

Quantile	Historical VaR	Normal distribution		t-distribution		t-copula & t-marginals	
		Mean	(Std)	Mean	(Std)	Mean	(Std)
0.01	489.93	397.66	(13.68)	464.66	(39.91)	515.98	(36.54)
0.025	347.42	335.28	(9.67)	326.04	(18.27)	357.40	(18.67)
0.05	270.41	280.69	(7.20)	242.57	(10.35)	260.27	(11.47)

Table 8: Mean and standard deviation of 100 VaR estimations (multiplied by 10^5) from simulated data following different distributions which are fitted to the data set D_1.

Quantile	Historical VaR	Normal distribution Mean (Std)	*t*-distribution Mean (Std)	*t*-copula & *t*-marginals Mean (Std)
0.01	155.15	138.22 (4.47)	155.01 (8.64)	158.25 (8.24)
0.025	126.63	116.30 (2.88)	118.28 (4.83)	120.08 (4.87)
0.05	98.27	97.56 (2.26)	92.35 (2.83)	94.14 (3.12)

Table 9: Mean and standard deviation of 100 VaR estimations (multiplied by 10^5) from simulated data following different distributions which are fitted to the data set D_2.

Quantile	Historical VaR	Normal distribution		t-distribution		t-copula & t-marginals	
		Mean	(Std)	Mean	(Std)	Mean	(Std)
0.01	183.95	156.62	(3.65)	179.18	(9.75)	179.41	(6.17)
0.025	141.22	131.54	(2.41)	124.49	(4.43)	135.21	(3.69)
0.05	109.94	110.08	(2.05)	91.74	(2.55)	105.67	(2.59)

Table 10: Mean and standard deviation of 100 VaR estimations (multiplied by 10^5) from simulated data following different distributions which are fitted to the data set D_3.

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