Inefficient School Choice in a Long-Run Urban Equilibrium

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September 2018

Abstract

We model centralized school matching as a second stage of a simple Tiebout-model and show that the two most discussed mechanisms, the deferred acceptance and the Boston algorithm, both produce inefficient outcomes, and that the Boston mechanism is more efficient than deferred acceptance. This advantage vanishes if the participants get to know their priorities before they submit their preferences. Moreover, the mechanism creates artificial social segregation at the cost of the disadvantaged if the school priorities are based on ex ante known (social) differences of the applicants.

JEL-Code: C78, H42, R23.

Keywords: School Choice, Deffered Acceptance, Boston Mechanism, Tiebout-hypothesis, social seggregation.

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1 Introduction

Matching matters. Not only marriage can make or ruin a man's life, but so can the choice of the wrong school, university, job or house. At first sight these matching problems seem very different in nature. While some of them, like most housing or job matching problems, are solved quite satisfactorily without regulatory interventions, others seem to inevitably generate chaotic conditions which cannot be eliminated without the help of a central matching authority.

The crucial difference between problematic matching problems and those solved satisfactorily in decentralized matching markets is the availability of sufficient flexible means to transfer the utility from one market side to the other. Thus we are faced with two separate branches of research on matching problems. On the one hand, there is a huge literature on matching in labor or housing markets, in which perfect information together with a flexible price system would allow an efficient perfectly competitive solution[1], while search or other frictions may distort the market outcome and hence call for political activities.[2] On the other hand, there is an unrelated literature on the allocation problems that arise in student or school matching problems, where there is no price mechanism available to organize the market.

Roth (1984) was the first to discuss the role of centralized authorities for the performance of the latter institutions. He described the problems caused by the unravelling of the search over time in the market for medical interns and attributed the success of the centralized matching procedure, the NRMP, which was introduced to solve the allocation problem, to the stability of the outcomes constructed by this algorithm. Later Roth (1991) confirmed this

[1]The linear assignment problem with a nice perfectly competitive solution was presented by Shapley and Shubik (1971). Roth and Sotomayor (1990, part III) present results from several papers which show that many of the properties of this special model generalize to models with non-linear transfers and many-to-one matching problems.

[2]As it is not the focus of this paper we do not try to give an adequate introduction to the literature on search and matching markets, but only refer to the scientific background paper "The Prize in Economic Sciences 2010 - Advanced Information" compiled by the Economic Sciences Prize Committee of the Royal Swedish Academy of Sciences.

intuition with the help of a comparison of seven British matching authorities showing that those which did not construct stable solutions but allowed simple manipulation by blocking pairs did not succeed to eliminated the chaotic search frictions. Following these seminal papers several authors identified further matching problems in which the lack of a price system caused various inefficiencies in markets with or without a central authority[3].

In this context also the school choice problem received considerable attention. The matching of pupils to schools is usually considered as an important political issue, because parents feel very strongly about the education of their children and articulate their disappointment about the outcome of a school matching very loud and clearly. Abdulkadiroğlu and Sönmez (2003) present an abstract formulation of the school choice problem and an algorithm to construct the efficient stable solution, while Abdulkadiroğlu et.al. (2005a, 2005b) describe and compare the performance of the two most popular school matching procedures used in the United States.

These papers are typical for the market design approach established in this branch of the matching literature[4]. They identify properties of a desirable solution, describe the procedures used to solve a real matching problem, compare the outcome of the real institution with the ideal solution, and propose mechanisms which would produce better outcomes. This way they helped to improve the performance of several matching markets including those which assign graduates of law, psychology or medicine to their first jobs, students and children to schools and universities, and even organ transplants to potential receivers.

However, concentrating on the last difficult step of these allocation prob-

[3]Roth and Xing (1994) present further matching problems which unravel over time. Unravelling problems in the market for law clerks are described in Avery et.al. (2001). In Germany the matching of medical students to universities is run by a complicated centralized matching procedure discussed in Braun et.al. (2010). Roth et.al. (2004,2005) even discuss the efficient assignment of kidney donors to transplant patients.

[4]Note that Roth (2008) does not only summarize the success story of these applications of game theory, but also mentions (p.8-9) the worries about the long run efficiency in the market for interns which we discuss below.

lems the analysis in all these models neglects that money would, in principle, allow to transfer utility also in these applications. In the market for medical interns or judicial clerks, for instance, transfer payments are available but not sufficiently flexible as a competitive price system would have to produce personalized wage offers based on information which is unavailable to the market participants. In other markets like those of the public education system competitive prices are considered as unfair and therefore as politically unacceptable. Even in marriage the distribution of the gains can be shared between the spouses with the help of explicit or implicit side payments.

Once we accept that, in principle, all real matching problems would allow some transfer of utilities the stability of the matching mechanism alone, that is for fixed transfers, is not as desirable as it appears, because in this case it only implies the ex post efficiency of a complicated rationing scheme. In a recent paper Azevedo and Leshno (2015) argue that stable solutions of a matching problem imply more than this ex post efficiency as they can be interpreted as competitive equilibria in which the quality expectations of the market participants on both sides of the market serve as a substitute for the price system. Indeed, this property of the stable solution can explain why many problems in decentralized markets, in particular the unravelling of the outcome over time, disappear if a stable matching mechanism is introduced, but it does not guarantee that the gains from trade realized in a matching market are distributed competitively, that is, without a distortion of long run incentives so that the overall efficiency of the allocation constructed by a stable matching mechanism does not follow.

Kamecke (1989), on the other hand, showed that the participation constraint in such a matching market may already be distorted if the transfer payments are determined before the efficient matching is implemented. Even if the subsequent paper by Bulow and Levin (2006) demonstrates that this inefficiency depends on the exact formulation of the matching model and that it almost vanishes if the model allows a little bit more competition, one would expect even stronger and more robust ex ante efficiency problems in matching situations in which there are no prices or fees available. A serious discussion of efficiency in matching markets should therefore include the distribution of long run rents into the analysis.

In this paper we address the question of the efficiency of school matching mechanisms. For this purpose we embed a standard school choice model in a standard location choice model. In section 2 we introduce a simple urban economy in which households consume land and a local public school service with rivalry in consumption. Our schools differ only by their location and the households prefer schools, because their children have to bear higher transportation costs if they live further away from their schools. The households choose their locations freely and the higher value of the locations close to the schools is reflected by higher equilibrium prices of these locations.

This simple economy with land and schooling has no natural source of market failure so that it is possible to decentralize the efficient allocation in a competitive market system. In section 3 we demonstrate that the long run efficient allocation is reached as a simple bid-rent equilibrium in the sense of Alonso (1964) if competitive prices can form for both school capacities and land. Moreover, we argue that we don't need school prices to reach this efficient solution, because the value of costless schooling is capitalized in the housing prices[5]. In the light of the Tiebout-hypothesis, it is therefore not surprising that the efficient allocation can still be reached if we allow a complicated location dependent housing price mechanism, or if we introduce a local authority which solves the school matching problem with the ex ante efficient location dependent matching mechanism.

In section 4 we will show that in this environment both, the stable deferred acceptance and the Boston matching mechanism produce inefficient allocations and that the Boston allocation is less inefficient than the stable

[5]Capitalization of local public services as a mode of the Tiebout (1956) hypothesis models was first proposed by Oates (1969), even though efficiency of these equilibria must be considered an exception as pointed out by Bewley (1981).

allocation. It is the central point of our paper that such inefficiencies appear as soon as the short run matching mechanism depends on the household preferences, and that these inefficiencies vary with the matching mechanism used. This is an important observation, because the school choice literature usually justifies the introduction of 'participatory' elements into the school matching mechanism by a reference to the desirable properties of a market mechanism. In the first paragraph of their seminal paper Abdulkadiroğlu and Sönmez (2003) state: *"Wealthy parents already have school choice, because they can afford to move to an area with good schools, or they can enroll their child in a private school. Parents without such means, until recently, had no choice of school, and had to send their children to schools assigned to them by the district"*. Then they concentrate on the discussion of criteria which a participatory matching mechanism should have such as Pareto efficiency, strategy proofness, envy freeness and – if the preferences of the schools are also considered – stability of the match. The result of their analysis is a recommendation how to introduce elements of household preferences into the school matching mechanism without violating fundamental requirements of fairness and efficiency, but this recommendation ignores the long run efficiency of, for instance, the residential housing market.

The reason for this market failure in our model is closely related to the discussion of cardinal utility considerations in school matching. Abdulkadiroğlu, Che and Yasuda (2011) demonstrate that there is a natural conflict between the strategy proofness of the deferred acceptance (DA) mechanism and the sensitivity to intensities of cardinal preferences which, for instance, the Boston mechanism (BM) incorporates better than deferred acceptence (DA). In the DA algorithm every household submits a preference ordering to the matching authority. The households are tentatively assigned to their first choices and if a school is over-demanded the households with the lowest priorities are rejected and have to turn to their next best alternatives.[6]

[6]If all schools rank the households in the same order the households with the highest

The tentative matching generates a Vickrey-property of this algorithm, because unsuccessful applicants can always turn to their next best alternatives later without diminishing their success probabilities for these alternatives by waiting in vain.

This is very different in the Boston mechanism (BM), where such a tentative matching is not used. Instead BM matches as many applicants as possible in each round and rejects only those in over-demanded schools who then have to apply for the remaining capacities. Consequently over-demanded schools may be filled already in the first round, creating an incentive to drop high ranked over-demanded schools from the list in order to improve the chance to get a place in a less over-demanded lower ranked school in an earlier round of the mechanism. This incentive to misrepresent preferences makes rational play in the BM very difficult and has therefore usually been considered as a major drawback of the BM.

Abdulkadiroğlu et.al. (2011) readdress this issue. They argue that strategy proofness means that the participants have no incentive to reveal the intensity of their preferences. They propose a model in which all households have the same ranking of schools but different cardinal von Neumann-Morgenstern utility values which they aggregate ex ante with the help of a symmetric priority lottery. In this model they show that the households with less intense preferences are the first to omit an over-demanded school from their list in the BM. This way they generate a framework in which ex ante efficiency distinguishes the matching mechanisms and makes BM perform better than DA even though all matchings are ex post efficient.

By embedding the school matching in a simple spatial model we provide an established economic interpretation for this argument. In the second stage of our model households have chosen their location and prefer closer schools because it is more costly for their children to travel from home to distant

priorities get their first choices, while households with lower priorities have to take what's left – so that they get exactly the same as in the serial dictatorship (if the same priorities are used). Pareto improvements for the households as computed by the top-trading cycles algorithm are not possible in our model.

schools. In the first stage the free mobility assumption generates an outcome which gives equal long run expected utility to equal households. Therefore transportation cost differences are capitalized in the equilibrium housing prices and unnecessary transportation costs hurt the equilibrium utility level, because housing revenues are lost. This way the aggregate transportation costs replace the cardinal utility measure employed by Abdulkadiroğlu et.al. (2011). As a consequence long run Pareto efficiency requires the minimization of total transportation costs so that applicants from a neighborhood with over-demanded schools should receive preferred access at the neighboring locations and we show in section 4, that this task is not solved by DA nor by BM, and that BM outperforms DA systematically.

The incentive to shade the individual school rankings, and hence the advantage of BM over DA, depends crucially on the uncertainty in the second stage of the model. In section 5 we will therefore turn to the question what happens to the market outcome in an environment with complete information, where the households know their types before they participate in the market mechanism. If the matching procedure ends with a lottery for overdemanded school capacities the uncertainty can easily be resolved if the lottery tickets are revealed to the households before they submit their preference rankings. We will first show that this information structure eliminates the incentive to reveal intensities so that both BM and DA produce the same allocation.

For these equivalent school choice mechanisms we will then prove a “Tiebout” (voting with the feet) segregation result in a model, where the households can predict the outcomes of the school choice already when they choose their housing location. We show that this information structure reestablishes efficiency at the cost of equitableness of the outcome. The privileged households receive an higher equilibrium utility because they pay lower rents in the neighborhoods with a larger excess demand for schools, where they can guarantee a place in the neighborhood school.

2 The model

In order to embed our school matching model in a simple general equilibrium framework we use the following spatial limit economy. A continuum of households live in a linear city with a continuum houses and S schools $s \in \mathcal{S}$. The quality of living is the same in every house and the quality of education is the same in every school so that location is the only characteristic that matters.

The set of households $\mathcal{H}$ is endowed with a probability measure μ, the house locations $l \in [0;1]$ with the Lebegues measure λ, and the schools located $L_s \in [0;1]$ with $L_1 < \ldots < L_S$ have capacities $K_s \in [0;1]$ with $\sum_s K_s = 1$. As all three sets have the same size it is possible to assign household $h \in \mathcal{H}$ to a house at the locations l_h and to a school $\sigma(h) \in \mathcal{S}$, such that every measurable $H \subset \mathcal{H}$ satisfies $\mu(H) = \lambda(l_H)$[7] and $\mu(H) \leq \sum_{s \in \sigma(H)} K_s$.

We assume that the school capacities are concentrated towards the right boundary of the town at $L = 1$. To define this concentration we denote the midpoints between two schools by $\overline{L}_s = (L_s + L_{s+1})/2$ for $s = 1, \ldots, S-1$ with $L_0 = \overline{L}_0 = K_0 = K_{S+1} = 0$ and $L_{S+1} = \overline{L}_S = 1$. Then we assume the relative schooling capacity in the rings $[\overline{L}_{s-1}; \overline{L}_s]$ is increasing as we move from the left city boundary $L = 0$ towards the right $L = 1$:

$$\frac{K_s}{\overline{L}_s - \overline{L}_{s-1}} < \frac{K_{s+1}}{\overline{L}_{s+1} - \overline{L}_s}. \tag{1}$$

Assumption (1) implies that the cumulated average excess demand at the ring boundaries $\left(\overline{L}_s - \kappa_s\right)$ is positive and n-shaped: it increases up to the location $\overline{s}$ at which $K_{\overline{s}} / \left(\overline{L}_{\overline{s}} - \overline{L}_{\overline{s}-1}\right) < 1 \leq K_{\overline{s}+1} / \left(\overline{L}_{\overline{s}+1} - \overline{L}_{\overline{s}}\right)$ and decreases thereafter. This way (1) generates excess demand for schooling at $L = 0$ and

[7]For measure preserving mappings as a natural generalization of a matching for a continuum of agents see Kaneko and Wooders (1986).

excess supply at $L = 1$.

Both houses and schools are public property. The houses are sold in a competitive market at prices $p(l)$, and the housing revenues $\int_0^1 p(l)\,dl$ together with a lump sum tax τ (or transfer if $\tau < 0$) constitute the public revenue which is used to finance schooling expenses. Schooling capacities are free of charge and they are assigned by a centralized matching authority.

Every household $h \in [0;1]$ has one child which has to travel from the family home at l_h to its school $\sigma(h)$. Households care about different locations of houses and schools only because this transportation to school is costly. In addition to the housing price $p(l_h)$ household h has to pay the linear transportation cost $t \cdot \left|L_{\sigma(h)} - l_h\right|$ for the distance from his home l_h to the location $L_{\sigma(h)}$ of his school $s(h)$ so that the total utility of h is given by

$$u_h = -\tau - p(l_h) - t \cdot \left|l_h - L_{\sigma(h)}\right|.$$

In order to organize the matching mechanism every household has a personal characteristic c_h which is distributed uniformly on $[0;1]$. For this characteristic we offer two interpretations. First, we may assume that education is costly and that c_h determines a linear cost γc_h needed to educate the child of household h so that school s has to bear the expenses $C_s = \gamma \cdot \int_{\sigma^{-1}(s)} c_h dh$. To cover these costs the government will assign a budget B_s to every school. Alternatively, we may assume that $\gamma = 0$ and that c_h is a lottery ticket which determines the priority of a household in the mechanism. The difference between these two interpretations is the question where the preferences of the schools for different households in the matching mechanism come from. In the first case ($\gamma > 0$) the schools rank the children according to their abilities c_h, in the second ($\gamma = 0$) the authority draws random lottery tickets c_h to determine this ranking. Both interpretations have applications in reality, and they do not alter the formal analysis.

Throughout the paper we will discuss a two stage model with a competitive housing market in the first and a school choice mechanism in the second stage. In the first stage the housing market clears at equilibrium prices $p^*(l)$, while the government determines the budgets B_s granted to the schools. In the second stage each household submits a complete ranking R_h from the set $\Re$ of complete ordered preference lists $R_h = (R_h(1), \ldots R_h(S))$ of all S schools[8]. These preferences together with the personal characteristics $(R, c) = (R_h, c_h)_{h \in [0;1]}$ – but not the household locations l_h – are then used by the central authority of type α to determine the matching $\sigma^\alpha(h) \in S$ of children to schools, while the government determines the lump sum tax (or subsidy) τ. Throughout the paper the (strategy) profile $R : [0;1] \to \Re$ is assumed to be measurable. The government avoids surpluses and deficits so that it selects the budget balancing $B_s = C_s$ in all schools, and the budget balancing lump sum tax (or transfer if $\tau < 0$) $\tau = \gamma/2 - \int_0^1 p(l)\, dl$.

The households always maximize their expected utilities. For this it is important to model the exact point in time, where the households observe their personal characteristic c_h. In section 4 we compare two models in which the households submit their rankings R without observing c_h. In section 5 we assume that the households know their c_h before they submit their rankings and (later) before they select their locations so that they can condition the corresponding decisions on c_h.

3 Efficiency

Our model is formulated such that the total education costs $\gamma/2$ are not influenced by the matching of children to schools, while the revenue from the housing market is redistributed to the households. Pareto efficiency therefore requires to select an aggregate transportation cost minimizing matching. In

[8]As the households do not have outside options it is reasonable to restrict the attention to complete rankings of all S schools. Under our assumption of sufficient capacity this guarantees that every ranking will generate a feasible solution under the matching algorithms discussed in the following.

our model with linear transportation costs and a capacity concentration at $L = 1$ this minimum is reached whenever no children travel unnecessarily to a school in the "wrong" direction towards $L = 0$. Denote the cumulative capacity by $\kappa_t = \sum_{s=0}^{t} K_s$ for $t = 0, \ldots, S$, then the "capacity ring" matching σ^* with

$$\sigma^* (h) = s \Leftrightarrow l_h \in [\kappa_{s-1}; \kappa_s] \tag{2}$$

obviously solves this problem. The corresponding minimal transportation costs are

$$T^* = t \cdot \sum_{s=1}^{S} \int_{\kappa_{s-1}}^{\kappa_s} |l - L_s| \, dl.$$

If all school locations s satisfy $\kappa_s \geq L_s$ then σ^* is the unique efficient matching. If there is some school s with $\kappa_s < L_s$ efficiency requires that K_s of the households in the ring $[\kappa_{s-1}; L_s]$ are assigned to school s, and that the rest travels to a school closer to $L = 1$. A matching which does not satisfy this condition is inefficient, but it does not matter for efficiency which children have to travel.

Since our simple urban model does not exhibit a natural market failure it is not surprising that the efficient allocations could be decentralized as competitive equilibria if location dependent prices $p^* (l)$ for the houses and cost-dependent school fees $F_s^* (c)$ were available. In this context the "law of one price" generalizes to the indifference conditions that all children are equally profitable for each school, and that all locations are equally good for equal households. Thus there must be a uniform profit π_s^* per child for each school s and an equilibrium utility

$$u_h^* = -\tau - F_{\sigma(h)}^*(c_h) - p^*(l_h) - t \cdot \left| l_h - L_{\sigma^*(h)} \right| .$$

such that the equilibrium school fees $F_s^*(c)$ for $s \in \{1, \ldots, S\}$ and the (continuous) equilibrium prices $p^*(l)$ satisfy

$$\begin{aligned} F_s^*(c) &= \pi_s^* + \gamma \cdot c \\ p^*(l) &= p_s^* - t\,|L_s - l| \qquad \text{for } l \in [\kappa_{s-1}; \kappa_s] \\ p_{s+1}^* - p_s^* &= \pi_s^* - \pi_{s+1}^* = 2 \cdot t \cdot \left(\overline{L}_s - \max\{\kappa_s, L_s\}\right). \end{aligned} \tag{3}$$

Notice that the changes of the housing prices $p^*(l)$ compensate the households for their travel expenses towards the center. Continuity of the housing prices is necessary because otherwise the expensive locations close to a discontinuity would not be demanded by the consumers. The housing prices at the school locations p_s^* must increase in s because $\kappa_s < \overline{L}_s$ so that there are always more households traveling towards the center, and these increases of the housing prices at the school locations must be offset by corresponding decreases of the school fees and profits so that the scarce schools at the outskirts make higher profits.

Finally note that the level of the equilibrium fees F_1^* and the housing prices p_1^* are undetermined, because higher fees and prices lead to lower τ and hence to the same allocation. The efficient equilibria are therefore characterized by (3), if we normalize, for instance, by $F_S^* = p_S^* = 0$. In such an equilibrium the households are assigned to locations and schools such that the total transportation costs are minimal, but the matching of household types c_h to locations is arbitrary.

For this perfectly competitive equilibrium to be feasible the households have to know their types c_h and the schools have to be allowed to discriminate their fees according to c. Throughout this paper we assume that at least the latter assumption is violated, but even if the schools have to implement uniform fees F_s^* for all children an efficient solution is still feasible as long as the uniform fees satisfy (3) and the schools are forced to accept all applicants. Of course, such a (restricted) equilibrium still requires differentiated school fees so that the assumption of free schooling makes it impossible.

However, even free schooling is still not enough to generate a market failure, because for $F_s = 0$ the value difference between different school neighborhoods capitalizes in the housing prices so that we could still reach an efficient competitive equilibrium if we could impose school dependent housing prices

$$\widehat{p}_s(l) = p^*(l) - F_s^*.$$

For these housing prices all households are indifferent so that the matching σ^* is an equilibrium outcome. For this competitive mechanism to work, however, the households have to pay one of the S different school dependent housing prices at each location.

So we eliminate this possibility as well and assume that there is only one housing price at each location – and can still not necessarily generate a market failure. If households have to pay the discontinuous prices

$$p^{**}(l) = p^*(l) - F_s^* \text{ for } \kappa_{s-1} \leq l \leq \kappa_s \text{ for all } s \in \{1, \ldots, S\}.$$

in the first stage of our model they would like to send their children to the closest school in the second stage, but if a central matching authority forces them to accept the efficient matching σ^* we would arrive again at the efficient equal treatment capitalization equilibrium. So when we now show that a preference depending matching mechanism generates inefficient outcomes, this demonstrates indeed that school choice introduces the second best problem into our model.

Notice that the mechanism which implements σ^* ex post has descriptive content as purely location dependent algorithms based, for instance, on school districts were and still are used in many real life school matching problems. Moreover, this mechanism would generate an ex post desire to send the children to other than the predetermined schools. This conflict between ex ante indifference between locations and ex post rationing, which can be interpreted as a major source for the strong movement in favor of a preference based school choice mechanism, is therefore exactly what we would expect in an efficient capitalization mechanism.

4 Matching under Uncertainty

In this section we discuss the efficiency of the two stage procedure if the households do not know their personal characteristic c_h when they submit their preferences. Consequently the households also have to select their locations without knowing c_h so that it is natural to assume that the location choices l_h are independent of the personal characteristics c_h such that the conditional distribution of the school qualifications c_h is uniform on $[0;1]$ for the subset of households h who live in a any measurable subset of locations $L \subset [0;1]$. Of course, the households know their locations in the second stage of the model so that the preferences submitted to the authority depend on l_h but not on c_h.

Free mobility then requires that the ex ante identical households expect the same utility level

$$u^* = -\tau - p^*(l) - t \cdot \int_0^1 \left| l - L_{\sigma(h)} \right| dc_h$$

at every location l if σ is the solution constructed by the matching authority in the second stage equilibrium of the model.

4.1 Deferred Acceptance

In this first subsection we assume that the centralized matching authority uses the deferred acceptance algorithm to construct a stable solution σ^S in the second stage of our model. In finite models the stability of a matching is a well-known concept. A proposed matching can be blocked by a school-household pair if the school prefers the household's child to some of its proposed pupils, while the household prefers the school to the proposed school. A matching is stable if it cannot be blocked, and it is well-known that such a stable outcome can be constructed with the following deferred acceptance (DA) algorithm[9]:

[9]See Roth and Sotomayor (1991) for a survey of the origins of the algorithm.

1. in the first iteration $i = 1$ every household applies for one of the places in the first school $R_h(1)$ on his preference list; for each school s the applicants are collected in the set $\mathfrak{A}^S_{s1}$.[10]

2. in iteration $i = 1, \ldots$ a critical cost level

$$c^S_{si} = \max\left\{c \in [0;1] \left|\mu(h \in \mathfrak{A}^S_{si} \left|c_h \leq c\right) \leq K_s\right.\right\}$$

   is determined for each school s; each school fills the open capacity K_s tentatively with the best households from the actual set of applicants $\mathfrak{A}^S_{si}$ and the remaining candidates in

$$\mathfrak{U}^S_i = \bigcup_{s=1\ldots S} \left\{h \in \mathfrak{A}^S_{si} \left|c_h > c^S_{si}\right.\right\}$$

   remain unmatched;

3. the unmatched households $h \in \mathfrak{U}^S_i$ were rejected in the present iteration i by some school which ranked j_{hi}-th on their preference list; in the next iteration these households are forwarded as additional applicants to the next schools $R_h(j_{hi} + 1)$ on their ranking lists so that

$$\mathfrak{A}^S_{si+1} = \left\{h \in \mathfrak{A}^S_{si} \left|c_h \leq c^S_{si}\right.\right\} \cup \left\{h \in \mathfrak{U}^S_i \left|R_h(j_{hi} + 1) = s\right.\right\}$$

   and the algorithm returns to step 2 setting $i = i + 1$; otherwise the algorithm stops with $c^S_{s\infty} = c^S_{si}$ and $\mathfrak{A}^S_{s\infty} = \mathfrak{A}^S_{si}$.

In the infinite model discussed in this paper stability is defined in the same way, and our version of DA works also for a continuum of households, even though it does not necessarily converge in finitely many steps. However, measurability of the strategies guarantees that the decreasing sequence of critical costs converge to $c^S_{s\infty} = \lim_{i\to\infty} c^S_{si}$ so that the algorithm converges to

[10] In the following the superscript S denotes stability while the subscript s is the variable for the schools.

a well-defined (tentative) stable limit[11] which assigns all applicants in $\mathfrak{A}^S_{s\infty} = \lim_{i\to\infty} \mathfrak{A}^S_{si}$ to school s so that the matching authority proposes $\sigma^S(h) = s$ for all $h \in \mathfrak{A}^S_{s\infty}$.

The structural assumption (1) makes it easy to predict the outcome of the deferred acceptance algorithm. For all households it is a dominant strategy to apply at the closest schools first and all schools use the same criterion c to rank their applicants, while (1) guarantees that for large H the schools are less over-demanded as we move towards the city center. We can therefore conclude that the first school $s = 1$ will eventually have the strictest entry requirement $c^S_{1\infty}$ and that schools further downtown accept also more and more costly applicants, $c^S_{s\infty} < c^S_{s+1\infty}$. Rejected applicants have to travel further downtown, and since the critical costs are decreasing a dominant strategy involves listing the closest school towards the city center next (perhaps also after listing some schools further away from the city center which can never be successful after a rejection). The new applicants allow the schools to improve the average cost parameter of their pupils, and again the children with the highest costs are rejected. This way we construct a class of dominant strategies which construct a unique ex post stable matching so that also the schools have no incentive to misrepresent their preferences.

Theorem 1 *(Stable matching) Suppose the households do not know their cost characteristic c_h when the application procedure starts, then in the second stage of the model it is a dominant strategy equilibrium if all households submit their true preferences. Moreover, every second stage equilibrium R^S strategies construct the stable matching*

$$\sigma^S(h) = \begin{cases} s & \text{if } \left\{l_h \in [\overline{L}_{s-1}; \overline{L}_s) \text{ and } c_h \in [0; c^S_{s\infty}]\right\} \\ & \text{or } \left\{l_h \in [0; \overline{L}_{s-1}] \text{ and } c_h \in (c^S_{s-1\infty}; c^S_{s\infty}]\right\} \end{cases}$$

[11] Note that each household can be rejected at most S times so that the total mass of rejections during the procedure is bounded by S. If a positive mass $\varepsilon > 0$ remained unmatched in the limit, the total mass of rejections would diverge.

where the critical cost level for school s is given by $c^S_{s\infty} = K_s / \left(\overline{L}_{s-1} - \kappa_{s-1}\right)$. The limit allocation constructed by DA in the two-stage model is inefficient.[12]

4.2 The Boston solution

Contrary to DA the Boston mechanism (BM) does not know a tentative matching. It immediately assigns best applicants to best schools so that households who are not successful with their earlier priorities are assigned to schools which are not filled in earlier iterations. Again the individual rankings R_h are selected from the set $\Re$ of all $\sum_{k=1}^{S} k!$ preference lists and R is a (measurable) strategy profile.

The Boston matching σ^B is constructed with a repetition of the following steps[13]:

1. in the first iteration $i = 1$ every household applies for one of the places in the first school $R_h(1)$ on his preference list; for each school s the applicants are collected in the set $\mathfrak{A}^B_{s1}$. The remaining capacity is set to $K^B_{s1} = K_s$.

2. in iteration $i = 1, \ldots, I$ each school s fills the remaining capacity K^B_{si} with the best households from the actual set of applicants $\mathfrak{A}^B_{si}$: for each school the critical cost level is determined as[14]

$$c^B_{si} = \begin{cases} 0 & \text{if } K^B_{si} = 0 \\ \sup\left\{c \,\middle|\, \mu(\left\{h \in \mathfrak{A}^B_{si} \,|\, c_h \leq c\right\}) < K^B_{si}\right\} & \text{if } K^B_{si} > 0 \text{ and } \mathfrak{A}^B_{si} \neq \emptyset \\ 1 & \text{otherwise,} \end{cases} \quad (4)$$

[12] All proofs are deferred to the appendix.

[13] The algorithm is the continuous generalization of the Boston matching scheme as presented, for instance, in Abdulkadiroglu et.al (2005, p.369).

[14] Note that c^B_{si} is defined such that single applicants (sets of measure zero) with a large c_h are not assigned to school s if the set of applicants just fits in the school, $\mu(\mathfrak{A}^B_{si}) = K^B_{si}$.

the low cost households are matched to school s, $\sigma^B(h) = s$ if $h \in \mathfrak{A}^B_{si}$ and $c_h \in [0; c^B_{si}]$, and the candidates in

$$\mathfrak{U}^B_i = \bigcup_{s=1\ldots S} \left\{ h \in \mathfrak{A}^B_{si} \left| c_h > c^B_{si} \right. \right\}$$

remain unmatched;

3. if there are unmatched households $h \in \mathfrak{U}^B_i$, they are forwarded as applicants to the next schools $R_h(i+1)$ on their lists so that

$$\mathfrak{A}^B_{si+1} = \left\{ h \in \mathfrak{U}^B_i \left| R_h(i+1) = s \right. \right\}$$

and setting $K^B_{si+1} = \max\left\{ K^B_{si} - \mu\left(\mathfrak{A}^B_{si}\right), 0 \right\}$ the algorithm returns to step 2; otherwise the algorithm stops with a feasible solution.

In every iteration of this algorithm every household is either matched or rejected so that the solution is constructed after at most S iterations. This computational simplification generates an efficiency gain, because the households now risk to lose an attractive alternative during the ongoing round so that they face a tradeoff if they rank over-demanded schools first. To see this consider the strategies of the households in the middle between two overdemanded schools. In terms of distance these households are indifferent between the two schools, but in BM both schools will be filled in the first round so that these households and their close neighbors should list the school at which they expect fewer applicants first. We can therefore expect that the households in the neighborhood of these midpoints are the first to turn to the schools toward the center of the city. This improves the performance of the matching mechanism, because the incentive to turn to a less over-demanded schools is stronger for those households who care less for the over-demanded school. BM will therefore generate a more efficient outcome than DA.

Unfortunately, the efficiency gain of the BM is not for free, because the finite Boston game is strategically much more complex than the dominance solvable stable mechanism. The game is not dominance solvable and the finite Boston game may suffer from a serious coordination problem which may lead to multiple inefficient equilibrium or to non-existence of a pure strategy equilibrium. In our simple limit model the latter problem disappears and all equilibria of the game will turn out to be well-structured, because all remaining strategic difficulties appear in the "Boston scramble", that is the strategic choice of the schools ranked second, third and so on, and the solution of this part of the mechanism is irrelevant for the overall welfare as the total transportation costs are not affected by the matching realized by the households who are rejected in the first round.

The following theorem shows that the Boston equilibrium exists, that it has the natural ring structure (as long as it is also imposed by efficiency) with increasing success probabilities towards the city center, and that the Boston allocation is in general less inefficient than the stable solution.

Theorem 2 *Suppose the households do not know their cost characteristic c_h when the application procedure starts, then the second stage of the model has a Boston equilibrium R^B. Moreover, all second stage Boston equilibrium strategies R^B generate the same critical cost levels $c^B_{11} < c^B_{21} < \ldots < c^B_{s^B 1} = c^B_{s^B+11} = \ldots = c^B_{S1} = 1$ and the corresponding ring borders b^B_s defined recursively by $b^B_0 = 0$ and $b^B_s = b^B_{s-1} + K_s/c^B_{s1}$ such that $(b^B_s - b^B_{s-1})$ of the households h living at locations $l_h \in \left[b^B_{s-1}; \max\{b^B_s; L_s\}\right]$ list school s first $(R^B_h(1) = s)$. The limit allocation constructed by BM in the two stage model is at least as efficient as that constructed by DA.*[15]

Notice that the ring structure implies in particular that in equilibrium every household h living at $l_h \in \left[b^B_{s-1}; b^B_s\right]$ ranks "his" school first, $R^B_h(1) = s$, if the Boston ring border is to the right of the school location, $b^B_s \geq L_s$. If this is not the case then the mass $\left(L_s - b^B_s\right)$ of households list a school further

[15] All proofs are deferred to the appendix.

downtown first, and the equilibrium indifference condition implies that it does not matter which of the households located in $[b_{s-1}; L_s]$ do this.

5 Early Information

In this section we give up the assumption that the households do not know their personal characteristic c_h before they submit their preferences or select their location so that the assignement of assignment of households to locations l_h and the households costs c_h may not be independent. We therefore discuss the outcome of the second stage for every (measurable) assignment l of houses to locations, and the stable school matching $\sigma^S(h \mid l)$ in this stage will also depend on this assignment.

The dominance argument of the previous section is not affected by this generalization so that DA still constructs this stable solution. The efficiency gain of BM, however, depends on the assumption that the households cannot foresee the consequences of their application decisions. If the households know the rankings – and hence the equilibrium construction of the algorithm – before they submit their preferences the critical cost levels are sufficient information to avoid unsuccessful applications so that every instability of the final outcome generates the potential for a successful deviation in the second stage. Consequently the stable matching is also reached in the BM equilibrium.

Theorem 3 *(Interim equivalence) Suppose all households know their characteristics* (l_h, c_h) *before the application procedure starts, then a BM equilibrium outcome coincides with the stable solution constructed by DA. Moreover, all households who are not assigned to school* S *under the stable matching list the school to which they are assigned in the stable matching first,* $h \notin \left(\sigma^S\right)^{-1}(S \mid l) \Rightarrow R_h^B(1) = \sigma^S(h \mid l)$.[16]

[16] All proofs are deferred to the appendix.

This equivalence result is interesting, because it brings back the established advantage of DA over BM. The two procedures yield equivalent outcomes, but DA constructs the stable solution from the preferences revealed truthfully by the households (and schools) BM requires that rational households compute (or learn) the critical cost levels before they submit their best replies.

We can push the same question one step further and ask what happens if the households have some information about the priorities c_h already when they chose their locations. This is particularly reasonable if these priorities are determined by the individual education costs, because the individual characteristics as well as the schools' preferences are to a large extent determined by the social attributes of the households so that it is not entirely convincing to assume that the preference of the schools come as a surprise. We will now show how the (equivalent) matching schemes DA and BM influence the final allocation if there is full information about the c_h from the very beginning of the two-stage procedure. For this we restrict our attention to equilibrium plans which satisfy "subgame perfection" in the sense that deviations in the first stage are evaluated under the assumption that the second stage matching generates a Nash equilibrium of the game.[17]

Under full information households with a large c_h know already in the first stage that they will not be successful in an over-demanded school so that these households have an incentive to move towards the city center. This diminishes the competitive pressure at the outskirts and makes the housing prices there interesting for households with low c_h. We can therefore expect a first stage sorting mechanism which drives the households into neighborhoods of similar types c_h.

Theorem 4 *(Ex ante social segregation) Suppose all households know their priorities c_h before the housing market clears, then BM as well as DA generate*

[17]Notice that this is not subgame perfection in the usual sense as our model combines a first stage competitive market with a second stage game.

an efficient equilibrium $\left(l^{FI}, \sigma^{FI}, p^{FI}\right)$ *in which lower priority households live closer to the center*[18] *and send their children to schools closer to the center*

$$c_h \in [\kappa_{s-1}; \kappa_s] \Rightarrow l_h^{FI} \in [\kappa_{s-1}; \kappa_s] \text{ and } \sigma^{FI}\left(h \mid l^{FI}\right) = s,$$

while the equilibrium prices are continuous and satisfy

$$\begin{aligned} p^{FI}(l) &= \phi_s - t \cdot |l - L_s| \text{ for } l \in [\kappa_{s-1}; \kappa_s] \\ \phi_{s+1} - \phi_s &= 2 \cdot t \cdot \left(\overline{L}_s - \max\{\kappa_s, L_s\}\right). \end{aligned}$$

All households whose children are assigned to the same school s *have the same equilibrium payment* $u_h^{FI} = -\phi_s - \tau$*, while households whose children are assigned to a school closer to the center are worse off,* $\phi_1 < \phi_2 < \ldots < \phi_S$*.*[19]

Comparing this equilibrium with the perfectly competitive outcome defined in (3) we see that the two competitive price functions p^* and p^{FI} coincide, and that both allocations are efficient. However, the segregation of types and the corresponding unequal treatment are new. Under perfect competition households have to pay their individual education costs γc_h so that households with a higher c_h are disadvantaged, but this unequal treatment vanishes as γ becomes small. Here the households with higher priorities in the school matching mechanism gain using their advantage to live in neighborhoods close to schools which do not accept children of disadvantaged households. In this sense our result can be interpreted as a a new type of market failure which reverts the Tiebout (1956) hypothesis where sorting by types is driven by efficiency in favour of all households.

6 Conclusion

In the light of the actual discussion on mechanism design and market engineering an economist who grew up when general equilibrium analysis was

[18]Notice that the equilibrium assignement l^{FI} inside the neighborhood rings $[\kappa_{s-1}; \kappa_s]$ is arbitrary and that the identity $l_h = c_h$ is one of them.

[19]All proofs are deferred to the appendix.

still an issue may get the impression that the literature focuses too much on the little problem at hand and too little on the context of these problems. In this paper we tried to demonstrate that this worry may not be entirely mislead. We tried to put the matching mechanism as one particular widely discussed rationing scheme into perspective by understanding it as last step of the solution to a larger economic allocation problem. In a simple urban context we could show that efficiency is indeed context dependent and that convincing criteria for a well-functioning matching mechanism may not be as convincing if we look at them from the contextual perspective.

We tried to make this point as simple as possible so that the model which we introduced for this purpose is too simplistic for a serious application. Nevertheless, we think that not only the central message of the paper is more general than the assumptions we impose, but that our approach also allows generalizations which have the potential to contribute to an interesting discussion of actual policy issues.

On the demand side our model oversimplifies the relevant characteristics of the school choice problem. In reality schools differ by their qualities and households have strong preferences for these differences. Our model allows to include this aspect into the willingness to pay for a location in the neighborhood of a certain school. Of course such an extension would complicate our analysis severely, but this exercise may contribute interesting new aspects to the ongoing discussion about the desirable properties of a school matching mechanism, and contrary to the recent attempts to identify further refinements which distinguish between different efficient outcomes for instance on the basis of cardinal utility or bounded rationality[20] our approach would allow a discussion solely on the grounds of established efficiency criteria.

On the supply side our assumptions are even further away from a serious attempt to describe reality, since our model assumes that school capacities are fixed, while both quality and quantity of schooling are clearly variable at least

[20]As, for instance, Abdulkadiroglu et.al. (2011) or Apesteguia and Ballester (2012).

in the long run. In order to reach an efficient solution in our model it would be enough to extend the schooling capacities at the outskirts and reduce them in the center of our town, and in the resulting model of horizontal quality differentiation there would be no serious matching problem left. However, if location is not the only quality characteristic of a school the supply of schools becomes a complicated positive and normative issue. A serious attempt to analyze the consequences of the generation and distribution of economic rents by an imperfect matching mechanism would have to embed the matching mechanism in a much more general Tiebout-type regional model, but this task is left for future research.

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8 Appendix: Proof of the Theorems

8.1 Stable Equilibrium (Theorem 1)

Dominance: Fix the strategies R_{-h} for all households except h. In our infinite model the critical costs $c^S_{s\infty}$ (with $c^S_{0\infty} = 0$) are not affected by the strategy R_h of household h so that the probability that h is assigned to the r-th ranked schools $R_h(r)$ in an arbitrary ranking R_h is given by

$$\pi_h(r, R_h) = \max_{s\in\{R_h(1)\ldots R_h(r)\}} c^S_{s\infty} - \max_{s\in\{R_h(1)\ldots R_h(r-1)\}} c^S_{s\infty}.$$

Suppose that household h ranks a school s on rank r even though it prefers s to the school $R_h(r-1)$. If this household exchanges the schools and submits

$$\widetilde{R}_h = (R_h(1), \ldots, R_h(r-2), R_h(r), R_h(r-1), R_h(r+1), \ldots, R_h(S))$$

instead of R_h, the total success probability of the schools $R_h(r-1)$ and $R_h(r)$ remains the same,

$$\begin{aligned}
\pi_h(r-1,\widetilde{R}) + \pi_h(r,\widetilde{R}) &= \max_{s\in\{\widetilde{R}_h(1)\ldots\widetilde{R}_h(r)\}} c^S_{s\infty} - \max_{s\in\{\widetilde{R}_h(1)\ldots\widetilde{R}_h(r-2)\}} c^S_{s\infty} \\
&= \max_{s\in\{R_h(1)\ldots R_h(r)\}} c^S_{s\infty} - \max_{s\in\{R_h(1)\ldots R_h(r-2)\}} c^S_{s\infty} \\
&= \pi_h(r,R) + \pi_h(r-1,R),
\end{aligned}$$

while the success probability of $\widetilde{R}_h(r-1) = R_h(r)$ satisfies

$$\begin{aligned}
\pi_h(r-1,\widetilde{R}_h) &= \max_{s\in\{\widetilde{R}_h(1)\ldots\widetilde{R}_h(r-1)\}} c^S_{s\infty} - \max_{s\in\{\widetilde{R}_h(1)\ldots\widetilde{R}_h(r-2)\}} c^S_{s\infty} \qquad (5)\\
&= \max\left\{0; c^S_{\widetilde{R}_h(r-1)\infty} - \max_{s\in\{\widetilde{R}_h(1)\ldots\widetilde{R}_h(r-2)\}} c^S_{s\infty}\right\} \\
&\geq \max\left\{0; c^S_{R_h(r)\infty} - \max_{s\in\{R_h(1)\ldots R_h(r-1)\}} c^S_{s\infty}\right\} \\
&= \pi_h(r,R_h)
\end{aligned}$$

with a strict inequality if $c^S_{R_h(r)\infty} > c^S_{R_h(r-1)\infty}$. We can therefore conclude that submitting the true R_h is at least as good and for some R_{-h} even better than any other strategy.

Stable Matching: Notice that $\pi_h(1,R_h)$ is always positive. It is therefore strictly dominated, and hence never a weak best reply, not to list the best school first, so that the households located at $l_h \in \left(\overline{L}_{s-1};\overline{L}_s\right)$ select school s as their first choices, $R^S_h(1) = s$. In the first iteration the critical cost levels are thus $c^S_{s1} = \min\left\{K_s/\left(\overline{L}_s - \overline{L}_{s-1}\right), 1\right\}$ so that monotonicity of the critical costs $c^S_{11} \leq \ldots \leq c^S_{S1}$ holds by (1) (with a strict inequality as long as $c^S_{s-11} < 1$).

In later rounds only rejected households $h \in \mathfrak{U}^S_1$ with $c_h < c^1_{s1}$ can be successful at school s and hence further decrease c^1_{s1}. As there are no such households for $s = 1$ we get $c^S_{11} = c^S_{1\infty} = K_1/\overline{L}_1 < c^S_{s\infty}$ for all $s > 1$, while it is strictly dominated, and hence never a weak best reply, for all h with $l_h \in [\overline{L}_0; \overline{L}_1)$ not to list school 2 as their second choice, so that an optimal ranking must satisfy $R^S_h(2) = 2$. As there are no further households $h \in \mathfrak{U}^S_1$ with $c_h < c^S_{21}$ we get $c^S_{22} = K_2/(\overline{L}_2 - K_1)$.

Iterating this argument for $i = 2 \ldots S$ we can conclude that there are no further households to diminish the critical cost for $s = i$ so that $c^S_{i1} = c^S_{i\infty} = K_i/\left(\overline{L}_i - K_{i-1} - \ldots - K_1\right) = K_i/\left(\overline{L}_i - \kappa_{i-1}\right) < c^S_{s\infty}$ for all $s > i$, while it is strictly dominated, and hence never a weak best reply, for all $h \notin \mathfrak{A}^S_{s1}$ with $l_h \in [\overline{L}_0; \overline{L}_i)$ not to list school $i+1$ before any school $s > i+1$, so that an optimal ranking must satisfy $R^S_h(r_1) = i+1 < s = R^S_h(r_2) \Rightarrow r_1 < r_2$. Again there are no further households $h \notin \bigcup_{s=i+1}^{S} \mathfrak{A}^S_{s1}$ with $c_h < c^S_{i+1i}$ so that we get $c^S_{i+1i+1} = K_{i+1}/(\overline{L}_2 - \kappa_i)$. This way we construct the critical costs $c^S_{s\infty}$ and the matching σ^S as defined in the theorem.

Inefficiency: Finally, note that (1) implies that $c^S_{1\infty} < 1$ so that there is some $\varepsilon > 0$ such that the mass ε of children travel from their locations $l \in \left(0; \varepsilon/\left(1 - c^S_{1\infty}\right)\right)$ to some school $s > 1$, while another mass ε of children travel from their locations $l \in \left(\overline{L}_1 - \varepsilon/c^S_{1\infty}; \overline{L}_1\right)$ to school 1. Moreover, we can select ε small enough to guarantee that $\overline{L}_1 - \varepsilon/c^S_{1\infty} > L_1$ so that the latter children live to the right of L_1. If these children exchange their schools the aggregate transportation costs are reduced by more than $\left(\overline{L}_1 - L_1 - \varepsilon/\left(2c^S_{1\infty}\right)\right) \cdot \varepsilon > 0$, because each such exchange of children saves the transportation costs spent on traveling away from the city center. ■

8.2 Boston Equilibrium (Theorem 2)

Throughout this section we denote a strategy profile in which household h plays R'_h instead of R_h by $(R \mid R'_h)$, the expected transportation costs of household h for a strategy profile R by $T_h(R)$, and the expected transportation cost conditional on $c \geq \bar{c}$ by $T_h^{c \geq \bar{c}}(R)$.

8.2.1 Properties of all Boston Equilibria

Excess supply for large s**:** If $c_{s1}^B = 1$ then all h with $l_h \in [\overline{L}_{s-1}; \overline{L}_s]$ select $R_h^B(1) = s$ so that $c_{s1}^B < 1$ whenever $K_s > \overline{L}_s - \overline{L}_{s-1}$, i.e. for all $s \leq \bar{s}$ by assumption (1). Since $\mu(\mathcal{H}) = 1$ there must be s with $c_{s1}^B = 1$ so let s^B be the smallest such s. Let $\widetilde{\mathcal{S}} = \{s \in \{s^B + 1, \ldots, S\} \mid c_{1s}^B < 1\}$, $\widetilde{\mathcal{S}}^c = \{s \in \{s^B + 1, \ldots, S\} \mid c_{1s}^B = 1\}$, and suppose that $\widetilde{\mathcal{S}} \neq \emptyset$ so that $s^B > \bar{s}$ implies that $\mu\left(\left\{h \mid R_h^B(1) \in \widetilde{\mathcal{S}}\right\}\right) > \sum_{s \in \widetilde{\mathcal{S}}} K_s$ even though $\sum_{s \in \widetilde{\mathcal{S}}} K_s > \sum_{s \in \widetilde{\mathcal{S}}} \overline{L}_s - \overline{L}_{s-1}$ by (1). This first round excess demand for $s \in \widetilde{\mathcal{S}}$ does not come from $l_h < L_{s^B}$, because a first round application $R_h^B(1) = s^B$ is always successful while a later ($r > 1$) application $R_h^B(r) < s^B$ never succeeds, so that a household with $R_h^B(1) \in \widetilde{\mathcal{S}}$ and $l_h < \overline{L}_{s^B}$ could reduce his expected transportation cost with a deviation to a strategy R'_h with $R'_h(1) = s^B$. The excess demand does also not come from neighborhoods of $s \in \widetilde{\mathcal{S}}^c$, since a household h with $l_h \in [\overline{L}_{s-1}; \overline{L}_s]$ for some $s \in \widetilde{\mathcal{S}}^c$ will always play $R_h^B(1) = s_h$. Consequently, $\left\{h \mid R_h^B(1) \in \widetilde{\mathcal{S}}\right\} \subset \cup_{s \in \mathcal{S}'} [\overline{L}_{s-1}; \overline{L}_s]$ and hence $\mu\left(\left\{h \mid R_h^B(1) \in \widetilde{\mathcal{S}}\right\}\right) \leq \sum_{s \in \widetilde{\mathcal{S}}} \overline{L}_s - \overline{L}_{s-1}$. This completes a contradiction.

Ring structure: Suppose there is an equilibrium in which households h_1 and h_2 located at $l_{h_1} = l_1 < l_2 = l_{h_2}$ list schools $R^B_{h_1}(1) = s_1$ and $R^B_{h_2}(1) = s_2$ first even though $L_{s_2} < L_{s_1}$. We show that h_1 does not lose deviating to h_2's strategy and that indifference $T_{h_1}\left(R^B\right) = T_{h_1}\left(R^B \mid R_{h_1} = R^B_{h_2}\right)$ is only possible in equilibrium if $l_1 < l_2 \leq L_{s_2} < L_{s_1}$.[21]

If $l_2 > \overline{L}_{s^B-1}$ the argument above implies that h_2 is assigned to his first ranked school at $L_{s_2} \geq L_{\overline{s}^B}$. $L_{s_1} > L_{s_2}$ implies $c_{s_1 1} = 1$ so that h_1 is assigned to s_1. As $l_1 < l_2$ a change to R_2 would increase h_1's payoff.

If $l_2 \leq \overline{L}_{s^B-1}$ both households have to travel towards the center if they are not successful with a first ranked s_1 or s_2 because $c_h > c^B_{s1}$ so that $T^{c_{s1}}_1 = T^{c_{s1}}_2 + t\left|l_2 - l_1\right|$. If in addition $l_1 \geq \left(L_{s_1} + L_{s_2}\right)/2$ then $c^B_{s_1 1} > c^B_{s_2 1}$ because the household h_2 could reduce the expected transportation costs by exchanging the ranks of the schools s_1 and s_2 in R^B_2 otherwise. Using this together with $\left|L_1 - l_2\right| = \left|l_2 - l_1\right| + \left|L_1 - l_1\right|$, the equilibrium condition $T_1\left(R^B\right) \leq T_1\left(R^B \mid R^B_2\right)$, and the triangle inequality $\left|L_2 - l_2\right| \leq \left|L_2 - l_1\right| +$

[21] In this part of the proof we use the index i instead of h_i, whenever this is possible without causing confusion.

$|l_2 - l_1|$ we derive the following contradiction:

$$\begin{aligned}
T_2\left(R^B\right) &= c^B_{s_21} t\,|L_1 - l_2| + \left(1 - c^B_{s_21}\right) T_2^{c \geq c^B_{s_21}}\left(R^B\right) \\
&= c^B_{s_21} t\left(|L_1 - l_1| + |l_2 - l_1|\right) + \left(1 - c^B_{s_21}\right)\left(T_1^{c \geq c^B_{s_21}}\left(R^B\right) - t\,|l_2 - l_1|\right) \\
&= \left(2c^B_{s_21} - 1\right) t\,|l_2 - l_1| + T_1\left(R^B \mid R^B_2\right) \\
&\geq \left(2c^B_{s_21} - 1\right) t\,|l_2 - l_1| + T_1\left(R^B\right) \\
&= \left(2c^B_{s_21} - 1\right) t\,|l_2 - l_1| + tc^B_{s_11}\,|L_2 - l_1| + \left(1 - c^B_{s_11}\right) T_1^{c \geq c^B_{s_11}}\left(R^B\right) \\
&\geq \left(2c^B_{s_21} - 1\right) t\,|l_2 - l_1| + tc^B_{21}\left(|L_2 - l_2| - |l_2 - l_1|\right) \\
&\quad + \left(1 - c^B_{s_11}\right)\left(T_2^{c \geq c^B_{s_11}}\left(R^B\right) + t\,|l_2 - l_1|\right) \\
&= \left(2c^B_{s_21} - 2c^B_{s_11}\right) t\,|l_2 - l_1| + T_2\left(R^B \mid R^B_1\right) \\
&> T_2\left(R^B \mid R^B_1\right)
\end{aligned}$$

If, on the other hand, $l_1 \leq \left(L_{s_1} + L_{s_2}\right)/2$ we use $|L_2 - l_1| = |l_2 - l_1| + |L_2 - l_2|$, the equilibrium condition $T_2\left(R^B \mid R^B_1\right) \geq T_2\left(R^B\right)$ and the triangle inequality $|L_1 - l_1| \leq |L_1 - l_2| + |l_2 - l_1|$ to get

$$\begin{aligned}
T_1\left(R^B\right) &= c^B_{21} t\,|L_2 - l_1| + \left(1 - c^B_{21}\right) T_1^{c \geq c^B_{21}}\left(R^B\right) \\
&= c^B_{21} t\left(|l_1 - l_2| + |L_2 - l_2|\right) + \left(1 - c^B_{21}\right)\left(T_2^{c \geq c^B_{21}}\left(R^B \mid R^B_1\right) + t\,|l_1 - l_2|\right) \\
&= T_2\left(R^B \mid R^B_1\right) + c^B_{21} t\,|l_1 - l_2| + \left(1 - c^B_{21}\right) t\,|l_1 - l_2| \\
&\geq T_2\left(R^B\right) + t\,|l_1 - l_2| \\
&= c^B_{11} t\left(|L_1 - l_2| + |l_1 - l_2|\right) + \left(1 - c^B_{11}\right)\left(T_2^{c \geq c^B_{11}}\left(R^B\right) + t\,|l_1 - l_2|\right) \\
&\geq c^B_{11} t\,|L_1 - l_1| + \left(1 - c^B_{11}\right) T_1^{c \geq c^B_{11}}\left(R^B \mid R^B_2\right) \\
&= T_1\left(R^B \mid R^B_2\right)
\end{aligned}$$

Since the equilibrium strategy minimizes transportation costs the inequalities must holds with "=". For $l_2 > L_1$ the triangle inequality and hence the second inequality is strict so that we must have $l_2 \leq L_1$ and the claim follows.

Monotonicity of c^B_{s1}: Suppose that $c^B_{11} \geq c^B_{21}$. Take a household h with $l_h \leq \overline{L}_1$ and denote the strategy which exchanges the ranking of schools 1 and 2 by $R^{1 \leftrightarrows 2}_h$. If $R^B_h(1) = 2$ then $R^{1 \leftrightarrows 2}_h$ satisfies $T^B_h\left(R^{1 \leftrightarrows 2}_h\right) \leq T^B_h(R)$ (with strict inequality if $l_h < \overline{L}_1$ or $c^B_{11} > c^B_{21}$), because $T^{c \geq c_{11}}_h\left(R^B \mid R^{1 \leftrightarrows 2}_h\right) = T^{c \geq c_{11}}_h(R) \geq |l_h - L_3|$, $T^{c \leq c_{21}}_h\left(R^{1 \leftrightarrows 2}_h\right) = |L_1 - l_h| \leq |L_2 - l_h| = T^{c \leq c_{21}}_h(R)$ and $T^{c_{11} \geq c \geq c_{21}}_h\left(R^{1 \leftrightarrows 2}_h\right) = |L_1 - l_h| \leq |l_h - L_3| \leq T^{c_{11} \geq c \geq c_{21}}_h(R)$. Therefore the ring structure property above implies that $R^B_h(1) = 1$ for all households h with $l_h < \overline{L}_1$. Since $c^B_{11} \geq c^B_{21}$ is only possible if there are at least as many first round applicants per place in school 1 as in school 2, assumption (1) implies that $\mu\left(\left\{h \mid R^B_h(1) = 2\right\}\right) \geq \overline{L}_2 - \overline{L}_1$. Thus there must be households h located at $l_h > \overline{L}_2$ with $R^B_h(1) = 2$. This, in turn, is only possible if $c^B_{21} > c^B_{31}$, because otherwise a deviation to $R^{2 \leftrightarrows 3}_h$ would be profitable for these households. Iterating this argument we get $c_{s1} > c_{s+11}$ for all s and hence $\sum_{s=1}^{S} \mu(A^B_{s1}) > \sum_{s=1}^{S} K_s = 1$, a contradiction to the assumption that the set of all households has measure 1.

Finally, suppose that $c^B_{11} < c^B_{21}$ and $c^B_{21} \geq c^B_{31}$. A repetition of the argument above for the second ring gives a contradiction unless $c_{21} = 1$, and the iteration of this argument completes the proof of our equilibrium properties.

Inefficiency: Monotonicity implies that the ring borders b^B_s satisfy $b^B_s \leq \overline{L}_s$ with a strict inequality if $c^B_{s+12} < c^B_{s+11}$ (i.e., for instance, if $s < s^B - 1$), because the household types $c \in \left(c^B_{s1}; c^B_{s+11}\right)$ have to travel further than to $s+1$ if they are not successful with $R^B_h(1) = s$, while the remaining types $c \in \left[c^B_{s+11}; 1\right]$ are also not better off after the first round rejection. In this case the solution is inefficient.

If $b_s^B < \overline{L}_s$ there is some $\varepsilon > 0$ such that households $h \in \left(\overline{L}_s - \varepsilon; \overline{L}_s\right)$ would have applied for school s under DA so that c_{s1}^S of them would travel outward to school s, while they travel to school $s+1$ towards the center under BS. Since all capacities are filled there are $\varepsilon \cdot c_{s1}^S$ households h with $l_h \leq b_s^B$ who are assigned to school s under BS, while they were assigned to school $s' > s$ under DA. This exchange of households saves at least the total transportation cost of $\Delta T = \varepsilon \cdot t \cdot c_{s1}^S \cdot \left(\overline{L}_s - b_s^B - \varepsilon\right)$.

8.2.2 Existence of a Boston Equilibrium

Fixed Point Argument: Throughout the following we restrict our search for an equilibrium strategy by reducing the complexity of potential strategies drastically. For $s < S$ we assume that all households in the ring $h \in [b_{s-1}; b_s]$ list school s first, that they list on the following ranks only the $S - s - 2$ schools s' which are located closer to but not at the center (i.e. $s < s' < S$), and that they list school S last. Moreover, we enumerate these $J_s = (S-s-1)!$ rankings of the form $\widetilde{R}_{sj} = \left(s, \widetilde{R}_{sj}(2), \ldots, \widetilde{R}_{sj}(S-s-1), S\right)$, and interpret ψ_{sj} as the share of households of every measurable subset of locations of $[b_{s-1}; b_s]$ who, independently of the cost characteristic c_h, submit the j-th of these modified (partial) rankings $\widetilde{R}_{sj}$.[22]

To apply a fixed point argument we construct a u.h.c. correspondence $(\beta, \psi) \to (B, \Psi)$ of right ring borders and of reduced best replies as follows. The vectors (β, ψ) and the subsets (B, Ψ) of potential right ring borders and modified partially mixed strategies are taken from the closed and convex set

$$\mathfrak{B} \times \mathfrak{P} = [0; \overline{L}_s]^{S-2} \times \left\{ p \in [0;1]^{(S-2)\times J_s} \left| \sum_{j=1}^{J_s} \psi_{sj} = 1 \right. \right\}.$$

[22] This way we avoid the introduction of the equivalent mixed strategies ψ_{sj} played by all households in the ring so that we don't have to modify the Boston algorithm.

For each $(\beta, \psi) \in \mathfrak{B} \times \mathfrak{P}$ we define the image $(B, \Psi)(\beta, \psi) \subset \mathfrak{B} \times \mathfrak{P}$ of our correspondence in three steps as follows.

First, we construct ring borders recursively by $b_1(\beta) = \beta_1$ and

$$b_s(\beta) = \max\{\beta_s; b_{s-1}(\beta)\} \text{ for } s = 1 \ldots S-2, \tag{6}$$

and define the expected transportation cost function for each ring $s < S$, for each vector of critical costs $c^B = \left(c^B_{si}\right)^{i<S}_{s<S}$, each mixed ranking $p_s \in \mathfrak{P}_s$ of the remaining schools $s+1 \ldots J_s$, and each location $l \in [0;1]$ as

$$T_s\left(c^B, p_s, l\right) = \sum_{j=1}^{J_s} p_{sj} \cdot t \cdot \left(\sum_{i=1}^{S-s} \left(c^B_{\widetilde{R}_{sj}(i),i} - c^B_{\widetilde{R}_{sj}(i-1),i-1}\right) \cdot \left|l - L_{\widetilde{R}_{sj}(i)}\right|\right)$$

with the understanding that $\widetilde{R}_{sj}(0) = 0$ and $c^B_{00} = 0$. It is obvious that T_s is continuous.

Second, we denote the critical costs constructed by the Boston algorithm according to (4) for the modified matching problem by $\widetilde{c}^B_{si}(\beta, \psi)$ and use this to define the strategy image of our fixed point correspondence for each s as the set of best replies at the right border

$$\Psi_s(\beta, \psi) = \arg\min_{p_s \in \mathfrak{P}_s} T_s\left(\widetilde{c}^B(\beta, \psi), p_s, \beta_s\right).$$

Third, we construct the set of locations at which the households are indifferent between listing s first and listing some other alternative s' first. For this purpose we define upper and lower contours of candidate locations for each s and each (β, ψ) by

$$\mathfrak{L}^{\leq}_s(\cdot) = \left\{l \in \left[0; \overline{L}_s\right] \mid T_s\left(\widetilde{c}^B(\cdot), \psi_s, l\right) \leq T_{s'}\left(\widetilde{c}^B(\cdot), \psi_{s'}, l\right) \text{ for all } s' > s\right\}$$
$$\mathfrak{L}^{\geq}_s(\cdot) = \left\{l \in \left[0; \overline{L}_s\right] \mid T_s\left(\widetilde{c}^B(\cdot), \psi_s, l\right) \geq T_{s'}\left(\widetilde{c}^B(\cdot), \psi_{s'}, l\right) \text{ for some } s' > s\right\}$$

and derive the corresponding set of potential borders for each ring s by

$$B_s(\beta, \psi) = \begin{cases} \mathfrak{L}^{\leq}_s(\beta, \psi) \cap \mathfrak{L}^{\geq}_s(\beta, \psi) & \text{if } \mathfrak{L}^{\leq}_s(\cdot) \cap \mathfrak{L}^{\geq}_s(\cdot) \neq \emptyset \\ \{0\} & \text{if } \mathfrak{L}^{\leq}_s(\beta, \psi) = \emptyset \\ \left\{\overline{L}_s\right\} & \text{if } \mathfrak{L}^{\geq}_s(\beta, \psi) = \emptyset. \end{cases} \tag{7}$$

Notice that the critical costs $\widetilde{c}_{si}^{B}(\beta,\psi)$ vary continuously with (β,ψ) since the number of applicants and hence of rejections in each iteration varies continuously with (β,ψ). Consequently, the strategy image $\Psi(\beta,\psi)$ is u.h.c. and convex valued. Notice further that the mean value theorem for the continuous T_s implies that $\mathfrak{L}_s^{\leqq}\cap\mathfrak{L}_s^{\geqq}\neq\emptyset$ if $\mathfrak{L}_s^{\leqq}\neq\emptyset$ and $\mathfrak{L}_s^{\geqq}\neq\emptyset$ so that $B_s(\beta,\psi)$ is well defined. By construction we have $L_{\widetilde{R}_{sj(i)}}>\overline{L}_s$ for all $i=2,\ldots$ so that

$$\begin{aligned} l<l'\leq\overline{L}_s \quad &\Rightarrow \quad \left|l-L_{\widetilde{R}_{sj(i)}}\right|=|l-l'|+\left|l'-L_{\widetilde{R}_{sj(i)}}\right|\Rightarrow \\ T_s\left(c^B,p_s,l\right)-T_s\left(c^B,p_s,l'\right) &= c_{s1}^{B}t\left(\left|l-\overline{L}_s\right|-\left|l'-\overline{L}_s\right|\right)+\left(1-c_{s1}^{B}\right)t\left(l'-l\right) \\ &\leq (l'-l)\cdot t. \end{aligned}$$

For $s'>s$ with $L_{s'}>\overline{L}_s$ the same argument gives

$$l<l'\leq\overline{L}_s\Rightarrow T_{s'}\left(c^B,p_{s'},l\right)-T_{s'}\left(c^B,p_{s'},l'\right)=t\cdot(l'-l).$$

Combining the two we get the monotone differences

$$T_s\left(c^B,p_s,l\right)-T_{s'}\left(c^B,p_{s'},l\right)\leq T_s\left(c^B,p_s,l'\right)-T_{s'}\left(c^B,p_{s'},l'\right) \tag{8}$$

so that there are $b_s^{\leqq}$ and $b_s^{\geqq}$ with $\mathfrak{L}_s^{\leqq}(\beta,\psi)=[0;b_s^{\leqq}]$ and $\mathfrak{L}_s^{\geqq}(\beta,\psi)=[b_s^{\geqq};\overline{L}_s]$ if the sets are non-empty. Consequently $B_s(\beta,\psi)\in\left\{\left[b_s^{\geqq};b_s^{\leqq}\right],\{0\},\left\{\overline{L}_s\right\}\right\}$ is convex valued. Moreover, $B(\beta,\psi)$ is u.h.c., because (8) guarantees that 0 is the last and $\overline{L}_s$ the first transportation cost minimizing candidate border, in the sense that $(\beta^n,\psi^n)\to(\beta,\psi)$ with $\mathfrak{L}_s^{\leqq}(\beta^n,\psi^n)=\emptyset$ and $\mathfrak{L}_s^{\leqq}(\beta,\psi)\neq\emptyset$ implies that $\mathfrak{L}_s^{\leqq}(\beta,\psi)=\{0\}\subset\mathfrak{L}_s^{\geqq}(\beta,\psi)$, while $(\beta^n,\psi^n)\to(\beta,\psi)$ with $\mathfrak{L}_s^{\geqq}(\beta^n,\psi^n)=\emptyset$ and $\mathfrak{L}_s^{\geqq}(\beta,\psi)\neq\emptyset$ implies that $\mathfrak{L}_s^{\geqq}(\beta,\psi)=\left\{\overline{L}_s\right\}\subset\mathfrak{L}_s^{\leqq}(\beta,\psi)$.

Since the mapping $(\beta,\psi)\to(B,\Psi)$ is convex valued and u.h.c there must be a fixed point $(\beta^*,\psi^*)\in(B,\Psi)(\beta^*,\psi^*)$ and it remains to show that this fixed point is a Nash equilibrium of the matching game.

Equilibrium To facilitate the notation let $c^*_{is} = c^B_{is}(\beta^*, \psi^*)$ and $T^*_s(l) = T_s(c^*, \psi^*_s, l)$. The ring borders have two important properties. First, they satisfy $b_s(\beta^*) = \beta^*_s$: if $b_s(\beta^*) = b_{s-1}(\beta^*) > \beta^*_s$ in (6) then $c^*_{s1} = 1$ (as there are no applicants for s); hence $(\overline{L}_{s-1}; \overline{L}_s) \cap \mathfrak{L}^{\geq}_s(\beta^*, \psi^*) = \emptyset \Rightarrow \beta^*_s = \overline{L}_s > \beta^*_{s'}$ for all $s' < s$. Secondly, the monotonicity argument used in the last paragraph of 8.1 is valid for β^*_s as well so that there must be a school $\overline{s}^*$ such that $c^*_{11} < \ldots < c^*_{\overline{s}^*1} = c^*_{\overline{s}^*+11} = c^*_{S-11} = 1$: if $c^*_{11} \geq c^*_{21}$ then $\beta^*_1 = \overline{L}_1$; assumption (1) implies that $\beta^*_2 - \beta^*_1 > \overline{L}_2 - \overline{L}_1$ so that $\beta^*_2 = \overline{L}_2$ and hence $c^*_{21} \geq c^*_{31}$; iterating this argument gives a contradiction to $c^*_{11} < 1$ so that $c^*_{11} < c^*_{21}$; if $c^*_{11} < c^*_{21}$ and $c^*_{21} \geq c^*_{31}$ we have $\beta^*_2 = \overline{L}_2$ and hence $c^*_{21} \geq c^*_{31}$; iterating this argument gives a contradiction unless $c^*_{21} = 1$, and so on.

We have to show that the deviation from the equilibrium strategy to some pure strategy R'_h of all schools does not pay for any household h located at $l_h \in [0; 1]$. To see this note first that ψ^*_s solves the optimization problem

$$\psi^*_s \in \arg\min_{p_s \in \mathfrak{P}_s} T_s(c^*, p_s, l_h)$$

for all households located at $l_h < \overline{L}_{\overline{s}^*}$: since all schools $s < \overline{s}^*$ are filled in the first round ($c^*_{si} = 0$ for all $i > 1$) households who are not successful with their first rank must travel to some $s \geq \overline{s}^*$; thus

$$\begin{aligned}
& T_s(c^*, \psi_s, l_h) \\
= \;& t \cdot c^*_{s1} \cdot |l_h - L_s| + t \cdot \sum_{j=1}^{J_s} \psi_{sj} \left(\sum_{i=2}^{S-s} \left(c^*_{\widetilde{R}_{sj}(i),i} - c^*_{\widetilde{R}_{sj}(i-1),i-1} \right) \cdot \left| L_{\widetilde{R}_{sj}(i)} - l_h \right| \right) \\
= \;& t \cdot c^*_{s1} \cdot |l_h - L_s| + (1 - c^*_{s1})(\beta^*_s - l_h) \\
& + t \cdot \sum_{j=1}^{J_s} \psi_{sj} \left(\sum_{i=2}^{S-s} \left(c^*_{\widetilde{R}_{sj}(i),i} - c^*_{\widetilde{R}_{sj}(i-1),i-1} \right) \cdot \left| L_{\widetilde{R}_{sj}(i)} - \beta^*_s \right| \right) \\
= \;& T_s(c^*, \psi_s, \beta^*_s) + t \cdot c^*_{s1} \cdot (|l_h - L_s| - |\beta^*_s - L_s|) + t \cdot (1 - c^*_{s1})(\beta^*_s - l_h)
\end{aligned}$$

implies that the difference $(T_s(c^*, \psi_s, l) - T_s(c^*, \psi_s, \beta_s^*))$ is independent of ψ_s so that ψ_s^* minimizes $T_s(c^*, \psi_s, l_h)$, too.

The fixed point strategies assign households $l_h \geq \overline{L}_{\overline{s}^*-1}$ to their closest schools so that these households cannot deviate successfully from their equilibrium strategies. The argument above implies that households who select $R_h'(1) = s' < \overline{s}^*$ as their first rank cannot do better than to continue with ψ_s^* (or one of the pure strategies which receive a positive probability in ψ_s^*). The optimality of ψ_s^* and the construction of β_s^* with the first case of (7) imply further that a household $l_h \in \left[\beta_{s-1}^*; \beta_s^*\right]$ can never gain from a deviation to $s' > s$ so that suppose that households h with $l_h \in \left[\beta_{s-1}^*; \beta_s^*\right]$ and $s < \overline{s}^*$ can deviate successfully to a strategy $R$with $R_h'(1) = s' < s$. In this case $l_h \in \mathfrak{L}_{s'}^{\geq}(\beta^*, \psi^*)$ so that there is some $s'' > s'$ with $T_{s''}(c^*, \psi^*, l_h) \leq T_{s'}(c^*, \psi^*, l_h)$ so that there is another successful deviation to some R'' with $R''(1) = s''$. Thus $s'' > s$ is not possible, while $s'' < s$ implies that there is another successful deviation to some R''' with $R'''(1) = s''' > s''$. Iterating this argument gives a successful deviation to a strategy $\widehat{R}$ with $\widehat{R}(1) = s$ and hence a contradiction to the optimality of ψ^*.■

8.3 Early Information

8.3.1 Interim Equivalence (Theorem 3)

Proof: Suppose the households apply such that the Boston mechanism generates an unstable solution, that is, there are households h with costs characteristic $c_h = c$ matched to school s and h' with cost characteristic $c_{h'} = c' > c$ matched to school s' such that $|L_s - l_h| > |L_{s'} - l_{h'}|$. Let household h change its ranking such that $\widetilde{R}_h(1) = s'$. Since h' is assigned to s' the school's capacity cannot be filled with better students in the first round (i.e. $c_{s1}^B \geq c'$), hence h' would be accepted in school s' and can thus increase his payoff with the deviation.

Suppose there is a household h located at $l_h < 1 - K_S$ with $\sigma^S(h) = s$ for some $s < S$ who does not list s first. As this household is eventually assigned to s there is empty capacity in school s after round 1, i.e. $c_{s1}^B = 1$. However, there are households h' located at $l_{h'} \in \left(\overline{L}_{s-1}; \overline{L}_s\right)$ whose $c_{h'}$ is so large that $\sigma^S(h') = S$. In equilibrium such a household h' must be assigned to school S so that it could deviate successfully from the equilibrium strategy listing school $s < S$ first.■

8.3.2 Social Segregation (Theorem 4)

Notice first that the full information prices defined by

$$p^{FI}(l) = \phi_s - t \cdot |l - L_s| \text{ for } l \in [\kappa_{s-1}; \kappa_s] \tag{10}$$

$$\phi_{s+1} - \phi_s = 2 \cdot t \cdot \left(\overline{L}_s - \max\{\kappa_s, L_s\}\right) \tag{11}$$

are continuous at the ring borders κ_s: assumption (1) implies $\overline{L}_s > \kappa_s$; for $L_s \geq \kappa_s$ (11) reduces to $\phi_{s+1} - \phi_s = t \cdot (L_{s+1} - L_s) > 0$ so that

$$\begin{aligned} \lim_{l \uparrow \kappa_s} p^{FI}(l) &= \phi_s - t \cdot (L_s - \kappa_s) \\ &= \phi_{s+1} - t \cdot (L_{s+1} - \kappa_s) = \lim_{l \downarrow \kappa_s} p^{FI}(l), \end{aligned}$$

while for $L_s < \kappa_s$ (11) becomes $\phi_{s+1} - \phi_s = t \cdot (L_{s+1} + L_s - 2 \cdot \kappa_s) = 2 \cdot t \cdot \left(\overline{L}_s - \kappa_s\right) > 0$ and so

$$\begin{aligned} \lim_{l \uparrow \kappa_s} p^{FI}(l) &= \phi_s - t \cdot (\kappa_s - L_s) \\ &= \phi_{s+1} - t \cdot (L_{s+1} - \kappa_s) = \lim_{l \downarrow \kappa_s} p^{FI}(l). \end{aligned}$$

Next we argue that $c_h \in [\kappa_{s-1}; \kappa_s] \Rightarrow l_h^{FI} \in [\kappa_{s-1}; \kappa_s]$ implies that the location choices l^{FI} generate the equilibrium allocation specified above. Since for all $s = 1 \ldots S$ the next best κ_s households live further away from the center, both DA and BM construct $\sigma^{FI}(h \mid \mathfrak{l})$. Moreover, no household has an incentive to deviate to another location: the households' payments $u_h^{FI} = -\phi_s - \tau$ coincide inside each school ring $[\kappa_{s-1}; \kappa_s]$, so that it is not possible to gain by choosing another location there; moving to a location in another ring and applying to the same school does not pay, because $p^{FI}(\cdot)$ is continuous with $\left|\left(p^{FI}\right)'(\cdot)\right| \leq t$; and moving to a location in another ring and applying to another school does also not pay, because applying to a school $s < \sigma^{FI}(h) - 1$ is not successful, while applying to a school $s > \sigma^{FI}(h)$ does not pay as $\phi_s > \phi_{\sigma^{FI}(h)}$.

To see that every equilibrium has these properties notice first that the argument of theorem 1 still applies so that every second stage equilibrium outcome constructs a stable solution of the school matching problem.

Free mobility implies that the equilibrium price function p^{FI} must be continuous, because otherwise a household could gain from changing to a cheaper location close to his equilibrium location and still being assigned to the same school. For the same reason the equilibrium payoffs of all households $h \in \left(\sigma^{FI}\right)^{-1}(s)$ assigned to the same school must coincide, $u_h^{FI} = -\phi_s - \tau$. Finally, this argument implies that $p^{FI}(l) - p^{FI}(L_s) = t\,|L_s - l|$ (and hence (10)) for every location $l \in [l_h; L_s]$ between L_s and a location l_h chosen as residence by some household $h \in \left(\sigma^{FI}\right)^{-1}(s)$, because otherwise either the households living close to l (if '>' holds) or the household living close to l_h (if '<' holds) could deviate successfully.

Take $s = S$, then by assumption (1) $\inf_{h \in (\sigma^{FI})^{-1}(S)}\{l_h\} \leq \kappa_{S-1}$, so that $p^{FI}(l) = \phi_S - t \cdot |l - L_S|$ holds for all locations $l \geq \kappa_{S-1}$. Households living at $l \in \left[\kappa_{S-1}; \overline{L}_{S-1}\right]$ would prefer $S-1$, so that $c_h \geq \inf_{(\sigma^{FI})^{-1}(S)}\{c_{h'}\}$ for all $h \in \left[\kappa_{S-1}; \overline{L}_{S-1}\right] \cap \left(\sigma^{FI}\right)^{-1}(S)$. Moreover, free mobility implies that this holds also for all other $h \in \left(\sigma^{FI}\right)^{-1}(S)$. Iterating this argument for

$s = S-1,\ldots,1$ gives the social segregation by schools, $h \in \left(\sigma^{FI}\right)^{-1}(s)$ and $h' \in \left(\sigma^{FI}\right)^{-1}(s-1) \Rightarrow c_h > c_{h'}$.

To derive the social segregation of neighborhood rings note that all households located at $l \geq \overline{L}_{S-1}$ are assigned to school S by DA so that these locations are occupied by $h \in \left(\sigma^{FI}\right)^{-1}(S)$. Suppose that all households h located at $l_h \geq \overline{L}_{\widehat{s}}$ are assigned to schools $s > \widehat{s}$ so that these schools have a total capacity of $\overline{L}_{\widehat{s}} - \kappa_{\widehat{s}} > 0$ left.

If $L_{\widehat{s}} < \kappa_{\widehat{s}}$ there must be $l, l' \in [L_{\widehat{s}}; \overline{L}_{\widehat{s}}]$ such that $h_l \in \left(\sigma^{FI}\right)^{-1}(\widehat{s})$ and $h_{l'} \in \left(\sigma^{FI}\right)^{-1}(s')$ for some $s' > \widehat{s}$. As $p^{FI}(\cdot)$ is decreasing up to l and increasing from l' onward there must be a boundary $b_{\widehat{s}}$ separating the two groups such that $l_h > b_{\widehat{s}} \Leftrightarrow \sigma^{FI}(h) > \widehat{s}$. As households located at $l < b_{\widehat{s}}$ are not assigned to a school $s > \widehat{s}$ we have $b_{\widehat{s}} = \kappa_{\widehat{s}} < \overline{L}_{\widehat{s}}$, and all households h located at $l_h \in \left[\overline{L}_{\widehat{s}-1}; \kappa_{\widehat{s}}\right]$ are assigned to school $\widehat{s}$ and $\phi_{\widehat{s}+1} - \phi_{\widehat{s}} = 2 \cdot t \cdot \left(\left(L_{\widehat{s}+1} + L_{\widehat{s}}\right)/2 - \kappa_{\widehat{s}}\right) = 2 \cdot t \cdot \left(\overline{L}_{\widehat{s}} - \kappa_{\widehat{s}}\right)$ (i.e.(11)).

If, on the other hand, $L_{\widehat{s}} \geq \kappa_{\widehat{s}}$ the interval $\left[L_{\widehat{s}}; \overline{L}_{\widehat{s}}\right]$ has not enough locations to host all households which have to be assigned to schools $s > \widehat{s}$. Thus there must be households $h \in \left(\sigma^{FI}\right)^{-1}(s)$ with $l_h \leq L_{\widehat{s}}$ for some $s > \widehat{s}$. Consequently $p^{FI}(\cdot)$ is increasing on $\left[\overline{L}_{\widehat{s}-1}; \overline{L}_{\widehat{s}}\right]$ with $h \in \left(\sigma^{FI}\right)^{-1}(\widehat{s}) \Rightarrow l_h \leq L_{\widehat{s}}$ so that $\phi_{\widehat{s}+1} - \phi_{\widehat{s}} = t \cdot \left(L_{\widehat{s}+1} - L_{\widehat{s}}\right) = 2 \cdot t \cdot \left(\overline{L}_{\widehat{s}} - L_{\widehat{s}}\right)$ (i.e.(11)) and $l_h \geq \overline{L}_{\widehat{s}-1} \Rightarrow \sigma^{FI}(h) \geq \widehat{s}$.■